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# On Extended Structures in Affine Toda Field Theory

by

Ulrich Karl Friedrich HARDER

A Thesis presented  
for the degree of  
Doctor of Philosophy

Department of Mathematical Sciences

The University of Durham  
May 1996



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## Abstract

Two areas of affine Toda field theory are explored in this thesis. First the introduction of a boundary into the real coupling affine Toda field theory. It has been shown by other authors that affine Toda field theory stays an integrable theory for certain boundaries. One such theory is the one corresponding to  $a_2^{(2)}$ . Its integrable boundary condition is described by two continuous parameters. Also, it is continuously connected to the natural Neumann condition, i.e. vanishing space derivative of the fields at the boundary. Classical reflection factors of incoming plane waves in the background of a static soliton solution are calculated for this theory. They fulfil a classical reflection bootstrap equation which is the classical limit of the reflection bootstrap equation for reflection matrices.

The second part is concerned with the  $a_n^{(1)}$  affine Toda field theory with imaginary coupling. The behaviour of oscillatory solitonic solutions, breathers is investigated. Explicit construction for breather solution are given. They originate from two soliton solutions. It is found that there are two different types of breathers depending on their constituent solitons. The constituent solitons are either of the same species or are anti-species of each other. Also, the topological charges of breather solutions are calculated and they are either zero or equal to a certain one soliton solution. These topological charges lie in the tensor product representation of the fundamental representations associated with the topological charges of the constituent solitons. The breather masses are, as expected, less than the sum of the masses of the constituent solitons.

## Preface

I would like to thank my supervisor Ed Corrigan for helpful discussions and encouragement in tackling the problems. Also, I would like to acknowledge Peter Bowcock, Patrick Dorey, Jens Gladikowski, Richard Hall, Meik Hellmund, Alexander Iskandar, Karen McGaul, William McGhee, Alistair MacIntyre, Niall MacKay, Gérard Watts, and Robert Weston for their comments, discussions and suggestions. I would like to thank the Commission of European Communities for a grant given under the Human Capital and Mobility programme (grant no. ERB4001GT920871). Außerdem möchte ich besonders meinen Eltern und meiner Schwester für Ihre Unterstützung während meines Studiums danken.

The material presented has not been submitted previously for any degree in either this or any other University. The thesis is based on research carried out between April 1993 and March 1996. The second chapter (from section 2.4) is based on work with my supervisor which has not been published yet. The third chapter is based on work with Alexander Iskandar and William McGhee and has been published in [95] and Alexander Iskandar's PhD thesis [101].

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# Chapter I

## Introduction into Affine Toda Field Theory

### 1.1 Introduction

Toda field theories are Lagrangian field theories characterised by Lie algebras. Their origin lies in early numerical work by Fermi, Pasta and Ulam. They investigated the behaviour of the energy of a one-dimensional dynamical system. In this system identical particles interacted with their nearest neighbours via a non-linear potential (spring)\*. Later Toda suggested that these systems might well be integrable if the potential is of an exponential nature [2]. This was subsequently proven to be true by Flaschka [3].

Toda field theory is a generalisation of the lattice theory. It comes essentially in three different flavours

- (1) (conformal) Toda field theory
- (2) affine Toda field theory
- (3) conformal affine Toda field theory.

In the following a short introduction into each of these theories will be given. Affine Toda field theory will get special attention as it is the main concern of this thesis. One should also mention that Toda field theory is connected to many other areas of mathematical physics. There is for example the Wess-Zumino-Novikov-Witten model. In [4] it is shown the the  $SL(2, \mathbb{R})$  WZNW model can be reduced to the Liouville theory. To show this one makes use of the Gauss decomposition of elements of  $SL(2, \mathbb{R})$  and the Polyakov-Wiegmann identity which allows one to express the WZNW action of a product of three group elements as the sum of their actions respectively modulo some local terms. As will be shown in the next section the Liouville theory is the “simplest” conformal Toda field theory and, as one would

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\* A modern review of the work can be found in[1]

expect, the reduction of the WZNW model can be conducted for other Lie algebras, and other conformal Toda field theories are recovered in this way.

The Ising model at criticality can be described by a conformal field theory with central charge  $c = \frac{1}{2}$ . Zamolodchikov [5] showed that the Ising model with an external magnetic field corresponds a perturbed conformal field theory which is an integrable model related to the Lie algebra  $e_8$ . Mansfield and Hollowood [6] showed that this integrable theory corresponds to the conformal Toda field theory associated with the Lie algebra  $e_8$  for a specific coupling constant. The affine Toda field theory associated with  $e_8$  can be seen as an integrable deformation away from the critical point.

More recently there has been some effort to investigate (2+1) dimensional models [7]. This thesis will deal only with two-dimensional theories.

## 1.2 Conformal Toda field theory

Conformal Toda field theory involves  $r$  scalar fields  $\Phi^i(x, t)$  written as

$$\Phi(x, t) = (\Phi^1(x, t), \dots, \Phi^r(x, t)).$$

The signature of the spacetime described by the variables  $x, t \in \mathbb{R}$  is Minkowskian  $(+, -)$ . Also, the following notation will be used throughout the thesis  $\partial_\mu = \frac{\partial}{\partial x^\mu}$ ,  $x^0 = t$ ,  $x^1 = x$ . The Lagrangian density of the theory is given by

$$\mathcal{L} = \frac{1}{2} \partial_\mu \Phi^a(x, t) \partial^\mu \Phi^a(x, t) - V(\Phi(x, t)). \quad (1.1)$$

The potential  $V$  distinguishes between the different theories by its relation to different Lie-algebras  $g^*$ . Each algebra  $g$  is characterised by its rank  $r$  and its simple roots  $\alpha_1, \dots, \alpha_r \in \mathbb{R}^r$ . The potential is

$$V(\Phi(x, t)) = \frac{m^2}{\beta^2} \sum_{i=1}^r e^{\beta \alpha_i \cdot \Phi(x, t)} \quad (1.2)$$

where  $m$  is the mass parameter and  $\beta$  the coupling constant. Classically the coupling constant is not important because it can be scaled away by defining  $\tilde{\Phi} = \beta \Phi$ . The

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\* For short introduction see appendix.



Lagrangian density is then

$$\mathcal{L} = \frac{1}{\beta^2} \left[ \frac{1}{2} \partial_\mu \tilde{\Phi}^a(x, t) \partial^\mu \tilde{\Phi}^a(x, t) - m^2 \sum_{i=1}^r e^{\alpha_i \cdot \tilde{\Phi}(x, t)} \right].$$

For the quantum theory the important quantity is  $\mathcal{L}/\hbar$  because in the path-integral formulation the integral is taken over  $\exp(i \int \mathcal{L}/\hbar)$  to determine vacuum expectation values. So, the classical limit of the quantum theory as  $\hbar \rightarrow 0$  corresponds to  $\beta \rightarrow 0$ , the weak coupling limit\*.

The equations of motion for (1.1) are

$$(\partial_t^2 - \partial_x^2) \Phi = \partial_\mu \partial^\mu \Phi(x, t) = -\frac{m^2}{\beta} \sum_{i=1}^r \alpha_i e^{\beta \alpha_i \cdot \Phi(x, t)}. \quad (1.3)$$

As explained later this theory is conformal and integrable. A simple example for this theory is the Liouville equation which corresponds to the  $g = su(2) = a_1$  theory. In this case the data given by the Lie-algebra is  $\alpha_1 = \sqrt{2}$ . The choice of  $m^2 = \beta = 1$  yields the equation in the form

$$\partial_\mu \partial^\mu \Phi(x, t) = -\sqrt{2} e^{\sqrt{2} \Phi(x, t)}.$$

If “ $x$ ” is fixed in the equations of motion (1.3) they are identical with the lattice models studied by Toda initially. Then  $\Phi_i = \alpha_i \cdot \Phi$  determines the displacement of the  $i$ -th mass on the lattice described by the Lie-algebra.

### 1.2.1 Conformal Invariance

To see the conformal invariance of the theory it is useful to introduce light cone coordinates

$$x_\pm = \frac{1}{\sqrt{2}}(t \pm x).$$

The equations of motions then are

$$\partial_+ \partial_- \Phi(x_+, x_-) = -\frac{m^2}{\beta} \sum_{i=1}^r \alpha_i e^{\beta \alpha_i \cdot \Phi(x_+, x_-)}. \quad (1.4)$$

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\* For more on this topic for the sine-Gordon theory, see chapter 6.4 of [8]

The differential operator is denoted by  $\partial_{\pm} = \frac{\partial}{\partial x_{\pm}}$ . Now a conformal transformation

$$x_{\pm} \rightarrow \bar{x}_{\pm}(x_{\pm})$$

will transform the lefthand side of equation (1.4) into

$$\partial_+ \partial_- \Phi(x_+, x_-) \rightarrow \bar{\partial}_+ \bar{\partial}_- \Phi(\bar{x}_+, \bar{x}_-) = \frac{\partial x_+}{\partial \bar{x}_+} \frac{\partial x_-}{\partial \bar{x}_-} \partial_+ \partial_- \Phi(x_+, x_-).$$

So, conformal invariance requires the field to transform in a particular way such that the right hand side will cancel the extra factor

$$\sum_{i=1}^r \alpha_i e^{\beta \alpha_i \cdot \Phi(x_+, x_-)} \rightarrow \frac{\partial x_+}{\partial \bar{x}_+} \frac{\partial x_-}{\partial \bar{x}_-} \sum_{i=1}^r \alpha_i e^{\beta \alpha_i \cdot \Phi(x_+, x_-)}. \quad (1.5)$$

If the field transforms as

$$\Phi(x_+, x_-) \rightarrow \bar{\Phi}(\bar{x}_+, \bar{x}_-) = \Phi(x_+, x_-) + \frac{\rho}{\beta} \ln \left( \frac{\partial x_+}{\partial \bar{x}_+} \frac{\partial x_-}{\partial \bar{x}_-} \right) \quad (1.6)$$

equation (1.5) is fulfilled if the vector  $\rho$  in (1.6) satisfies

$$\rho \cdot \alpha_i = 1 \quad i = 1, \dots, r,$$

and can therefore be expressed in terms of fundamental weights  $\lambda_i$

$$2\lambda_i \frac{\alpha_j}{|\alpha_j|^2} = \delta_{ij} \Rightarrow \rho = \sum_{i=1}^r \frac{2\lambda_i}{|\alpha_j|^2}.$$

A more detailed and in depth discussion of this issue can be found in [9-11]. The quantisation of the conformal Toda field theory gives a coupling dependent representation for the Virasoro algebra for the *ade* series [9-11]

$$c(\beta) = r + 48\pi |\rho|^2 \left( \frac{\beta}{4\pi} + \frac{1}{\beta} \right)^2. \quad (1.7)$$

This formula reveals a symmetry of the quantum theory under the transformation  $\beta \rightarrow \frac{4\pi}{\beta}$  which is not present in the classical theory.

### 1.2.2 Integrability

For a finite dimensional Hamiltonian system integrability means that there are  $N$  conserved charges  $K_i$  for a system with a  $2N$  dimensional phase space. These charges have to be in involution, i.e.  $\{K_i, K_j\} = 0$  ( $\{\cdot, \cdot\}$  denotes the Poisson bracket). Also, they are functionally independent. Liouville showed that the evolution of these systems can be completely determined in principle. However in practice there may be problems to do this explicitly. A field theory with Hamiltonian description is covered by this theorem when it is generalised to an infinite dimensional system. Integrability in this case requires the existence of infinitely many conserved charges which are again in involution and functionally independent. A good example in which the charges can be constructed explicitly is the KdV equation. Most text books, e.g.[12], on solitons give a derivation of the charges.

For affine Toda field theory one can show the existence of infinitely many independent conserved charges, which are in involution, once the Lax-pair is known. Also, one can work out solutions of the conformal Toda field theories. This was first done by Leznov and Saveliev[13]. A good review article about how to find solutions is given by Olive [14]. For the affine Toda field theory the integrability shall be examined in a little more detail in the next section.

## 1.3 Affine Toda field theory

Affine Toda field theory [15-19] is a generalisation of the Toda field theory which has been described in the previous section. Though it will be shown that the conformal invariance of the theory is not preserved, the integrability survives. It can be classified as a perturbed conformal field theory [6,20]. The study of affine Toda field theory is usually divided into two different regimes of the coupling constant, the real and the imaginary one. More precisely this corresponds to either real fields or complex fields as solutions. For real values of the coupling constant there exists a particle spectrum which has been studied extensively [21-34] The classical mass spectrum and the  $S$ -matrices of the quantum theory will be reviewed later. For the imaginary coupling constant (complex affine Toda field theory)[35] the theory possesses a spectrum of solitons. Again these soliton solutions have been examined

by many authors [35-40]. Efforts to determine  $S$ -matrices for complex affine Toda field theory have been made [41-44]. This will be reviewed later. In the case of the *ade* series the masses of solitons and particles corresponding to the same node of the Dynkin diagram are linearly related for theories associated with the same Lie-algebra [35,45].

### 1.3.1 The Origin of affine Toda Field Theory

A problem of the conformal Toda field theory is that the potential has its only minimum for  $\Phi_i \rightarrow -\infty$  i.e.  $e^{\beta\Phi_i} = 0$ , where  $\Phi_i = \alpha_i \cdot \Phi$ . This behaviour can be seen as a hint for the conformal invariance of the system. To get a stable point for a finite field and to preserve the integrability of the theory one perturbs the potential (1.2) by

$$\delta(V(\Phi)) = \frac{\epsilon m^2}{\beta^2} \exp[\beta\alpha_0\Phi]$$

where  $\alpha_0$  is an additional (“affine”) root such that  $\sum_{i=0}^r n_i \alpha_i = 0$ . The Dynkin indices  $n_i \in \mathbb{N}$  depend on the algebra  $g$  [46]. This yields the potential of affine Toda field theory

$$V = \frac{m^2}{\beta^2} \sum_{i=1}^r \exp[\beta\Phi_i] + \frac{\epsilon m^2}{\beta^2} \exp[\beta\alpha_0 \cdot \Phi_i].$$

This potential has the minimum  $\Phi^{(0)}$

$$\sum_{i=1}^r \alpha_i \exp[\beta\alpha_i\Phi^{(0)}] = -\epsilon \alpha_0 \exp[\beta\alpha_0\Phi^{(0)}] \quad (1.8)$$

multiplying (1.8) by  $\alpha_j$  and using the matrix  $\hat{C}_{ij} = \frac{1}{2}\alpha_i^2 C_{ij}$ , which is a conveniently rescaled Cartan matrix, (1.8) relationship implies

$$\exp[\beta\alpha_i\Phi^{(0)}] = -\epsilon \hat{C}_{ij}^{-1} \alpha_j \alpha_0 \exp[\beta\alpha_0\Phi^{(0)}].$$

So a shift in the field by  $\Phi^{(0)}$ ,  $\Phi = \phi + \Phi^{(0)}$  yields the potential

$$V(\phi) = \frac{m^2\epsilon}{\beta^2} \exp[\beta\alpha_0\Phi^{(0)}] \left[ \exp[\beta\alpha_0\phi] - \sum_{i,j=1}^r \exp[\beta\alpha_i\phi] (\hat{C}^{-1})_{ij} \alpha_j \alpha_0 \right].$$

The potential can now be written as

$$V(\Phi(x, t)) = \frac{m^2}{\beta^2} \sum_{i=0}^r n_i (e^{\beta\alpha_i \cdot \Phi(x, t)} - 1) \quad (1.9)$$

where the mass  $m^2$  has been redefined as  $\frac{m^2\epsilon}{\beta^2} \exp[\beta\alpha_0\Phi^{(0)}]$ . Subtracting  $-1$  ensures that the potential vanishes for  $\Phi = 0$ . The Lagrangian density of affine Toda field theory is

$$\mathcal{L} = \frac{1}{2} \partial_\mu \Phi^a(x, t) \partial^\mu \Phi^a(x, t) - V(\Phi(x, t)). \quad (1.10)$$

The equations of motion are

$$\partial_\mu \partial^\mu \Phi(x, t) = -\frac{m^2}{\beta} \sum_{i=0}^r n_i \alpha_i e^{\beta \alpha_i \cdot \Phi(x, t)} \quad (1.11)$$

### 1.3.2 Conformal Invariance

By comparison with section (1.2.1) where the conformal invariance of the conformal Toda field theory has been shown it becomes obvious that affine Toda field theory is not conformally invariant. Because  $\sum_{i=0}^r n_i \alpha_i = 0$  implies that for the vector  $\rho$  in (1.6)

$$\rho \cdot \alpha_0 = -\frac{1}{n_0} \sum_{i=1}^r n_i \neq 1$$

which makes it impossible to fulfil (1.5) after the transformation (1.6) of the field.

This problem can be overcome by a redefinition of  $\rho$ . It leads to the conformal affine Toda field theory which, as its name suggests, possesses conformal invariance. Instead of taking values in the Lie algebra  $\mathfrak{g}$  the field in the conformal affine Toda field theory takes its values in the Cartan sub-algebra of the associated affine Lie algebra  $\hat{\mathfrak{g}}$  [46]. In terms of the Cartan sub-algebra the field can be written as

$$\Phi = \phi \cdot H + \xi k + \eta d'$$

where  $H$ ,  $k$  and  $d'$  are explained in the appendix. The equations of motions are

$$\partial^2 \Phi + \frac{4\mu^2}{\beta} \sum_{i=0}^r n_i H^{\alpha_i} e^{\beta \alpha_i \cdot \Phi} = 0$$

where  $[E^{\alpha_i}, E^{\alpha_j}] = \delta_{ij} H^{\alpha_i}$  in the Chevalley basis. This theory has been investigated by Bonora [47,48] and Aratyn et al. [49-52]. It has been useful in the algebraic solution of affine Toda field theory [45]. Its disadvantage is that its scalar fields are no longer in Euclidean space and therefore the energy is not positive definite. Its restriction to Euclidean space breaks the conformal invariance and introduces a mass scale. The result is again the affine Toda field theory.

### 1.3.3 Integrability of affine Toda field theory

To establish the integrability of affine Toda field theory it is useful to examine the Lax pairs or the zero curvature condition [15-17,53] which can be stated as:

$$F_{01} = \partial_0 A_1 - \partial_1 A_0 + [A_0, A_1] = 0.$$

If the two components of the two-dimensional vector potential  $A_\mu$  are written as follows one can retrieve the Toda equations

$$\begin{aligned} A_0 &= \frac{1}{2} H \cdot \partial_1 \Phi + \sum_{i=0}^r m_i (\lambda E_{\alpha_i} - \frac{1}{\lambda} E_{-\alpha_i}) e^{\alpha_i \cdot \Phi/2} \\ A_1 &= \frac{1}{2} H \cdot \partial_0 \Phi + \sum_{i=0}^r m_i (\lambda E_{\alpha_i} + \frac{1}{\lambda} E_{-\alpha_i}) e^{\alpha_i \cdot \Phi/2}. \end{aligned}$$

The operators  $H_i$ ,  $E_{\alpha_i}$ ,  $E_{-\alpha_i}$  are the usual notation for the Cartan sub algebra corresponding to simple and affine roots of the Lie algebra  $g$  (for their definition see appendix). The spectral parameter is  $\lambda$  and the  $m_i$ 's satisfies

$$m_i^2 = n_i \frac{\alpha_i^2}{8}.$$

The classically unimportant parameters  $m$  and  $\beta$  have been scaled away. Define a path ordered integral by

$$T(a, b; \lambda) = P \exp \int_a^b dx^1 A_1.$$

It satisfies formally

$$\frac{d}{dt} T = T A_0(b) - A_0(a) T$$

and therefore the quantity  $Q(\lambda)$

$$Q(\lambda) = \text{tr} T((-\infty, \infty); \lambda)$$

is time-independent when  $\partial_1 \Phi \rightarrow 0$  as  $|x^1| \rightarrow \infty$  and also

$$\Phi(\infty) = \Phi(-\infty) + 2\kappa, \quad (\kappa \cdot \alpha_i) \in \mathbb{Z}.$$

The Lax pair allows a gauge transformation after which the potentials  $A_\mu$  lie in the Cartan sub algebra:

$$A_1 \rightarrow a_1 = \lambda E_1 + \sum_{s \geq 1} \lambda^{-s} h_s I_0^{(s)} \quad (1.12)$$

where  $E_{\pm} = \sum_{i=0}^r m_i E_{\pm\alpha_i}$  and  $s$  are the exponents of the Lie algebra modulo  $h$ ,  $h_s = h_{s+n}$ . Now the zero-curvature condition simplifies to

$$\partial_0 a_1 = \partial_1 a_0.$$

Which implies that the integral of  $a_1$  is conserved over the whole line. Also, as the choice of  $\lambda$  was arbitrary there are infinitely many conserved quantities  $Q_s$

$$Q_s = \int_{-\infty}^{\infty} dx^1 I_0^{(s)}.$$

It can be shown from (1.12) that  $\lambda$  scales under a Lorentz transformation  $\lambda \rightarrow l\lambda$  such that the light cone components of the potentials transform correctly. The conserved quantities scale with a factor  $l^s$ . One also needs to show that the conserved quantities are in involution. This is done by showing the existence of a classical  $r$ -matrix for which

$$[T(\lambda) \otimes T(\mu)] = [r(\lambda/\mu), T(\lambda) \otimes T(\mu)]$$

where  $T(\lambda) = T(-\infty, \infty; \lambda)$ . Further details can be found in [19].

### 1.3.4 Examples of affine Toda field theories

For the affine Toda field theory the simplest example is that related to the extended  $su(2)$  algebra,  $a_1^{(1)}$ . Here  $n_0 = 1$ ,  $n_1 = 1$ ,  $\alpha_0 = -\alpha_1 = -\sqrt{2}$ , the resulting equation is the Sinh-Gordon equation

$$\begin{aligned} \partial_{\mu} \partial^{\mu} \Phi(x, t) &= -\frac{m^2}{\beta} \sqrt{2} (e^{\sqrt{2}\beta\Phi(x,t)} - e^{-\sqrt{2}\beta\Phi(x,t)}) \\ &= -\frac{m^2}{\beta} 2\sqrt{2} \sinh \sqrt{2}\beta\Phi(x, t). \end{aligned} \quad (1.13)$$

Another example is  $a_2^{(2)}$  for which  $n_0 = 2$ ,  $n_1 = 1$ ,  $\alpha_1 = \sqrt{2}$ ,  $\alpha_0 = -\frac{1}{\sqrt{2}}$ . The equation of motion for this system is

$$\partial_{\mu} \partial^{\mu} \Phi(x, t) = -\frac{\sqrt{2}m^2}{\beta} (e^{\beta\sqrt{2}\Phi(x,t)} - e^{-\beta\frac{1}{\sqrt{2}}\Phi(x,t)}). \quad (1.14)$$

This equation is generally known as the Bullough-Dodd [54] or Jiber-Shabat [55] equation even though it has been known long before these publications. In 1910 Tzitzéica [56] mentions the equation in connection with the geometry of surfaces as Habibullin [57] has pointed out.

### 1.3.5 Duality

The Dynkin diagrams of affine Lie algebras fall into two categories under the transformation of the roots

$$\alpha_i \rightarrow 2 \frac{\alpha_i}{|\alpha_i|^2}.$$

First there is the set of theories which have roots of equal length,  $a_n^{(1)}$ ,  $d_n^{(1)}$ ,  $e_n^{(1)}$  and the only one with roots of three different lengths  $a_{2n}^{(2)}$ . They are mapped onto themselves by the transformation and called self-dual. Secondly the remaining theories are mapped into each other by the transformation and come in dual pairs  $(b_n^{(1)}, a_{2n-1}^{(2)})$ ,  $(c_n^{(1)}, d_{n+1}^{(2)})$ ,  $(g_2^{(1)}, d_4^{(3)})$  and  $(f_4^{(1)}, e_6^{(2)})$ . In quantum affine Toda field theories there is a transformation mapping the coupling constant  $\beta \rightarrow \frac{4\pi}{\beta}$ . The quantum theories of self-dual theories are unchanged under the transformation. The situation for the quantum theories corresponding to those algebras which come as dual pairs appears to be more complicated. There is one quantum theory for each dual pair. A theory corresponding to, for instance  $(g_2^{(1)}, d_4^{(3)})$ , will have its approximation as  $\beta \rightarrow 0$  provided by the theory associated with  $g_2^{(1)}$  whereas the approximation as  $\beta \rightarrow \infty$  is given by the  $d_4^{(3)}$  theory [32]. The transformation of the coupling constant effectively implements the mapping of the roots.

### 1.3.6 Folding

Due to the symmetry of some Dynkin diagrams there is a connection between the simply laced and non-simply laced theories. The procedure connecting them is known as folding and was introduced by Olive and Turok [18]. One can for instance fold  $d_4^{(1)}$  into  $a_2^{(2)}$ . For this one has to identify the roots  $\alpha_i$ ,  $i = 0, \dots, 3$ , of  $d_4^{(1)}$  with those of  $a_2^{(2)}$   $\alpha'_j$ ,  $j = 0, 1$ , in the following way

$$\alpha'_0 = \frac{1}{4}(\alpha_0 + \alpha_1 + \alpha_3 + \alpha_4) \text{ and } \alpha'_1 = \alpha_2.$$

If the  $\tau$ -functions (a concept that will be explained later) of  $\alpha_0, \alpha_1, \alpha_3, \alpha_4$  are the same, which they are for a special case, then  $\alpha'_0$  has the same  $\tau$ -function as  $\alpha_0$  and  $\alpha'_1$  the same as  $\alpha_2$ .



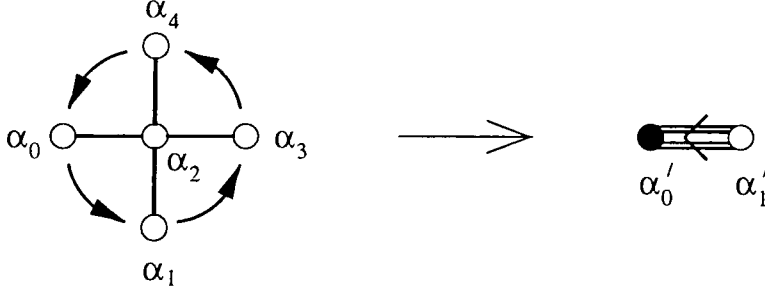


Fig. 1.1: Folding of  $d_4^{(1)}$  to  $a_2^{(2)}$

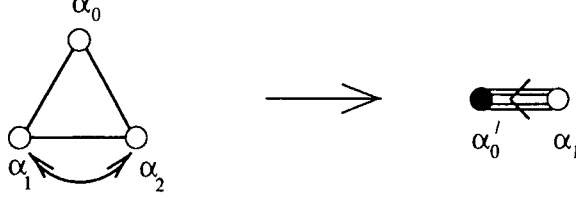


Fig. 1.2: The folding of  $a_2^{(1)}$  into  $a_2^{(2)}$

Alternatively one can fold  $a_2^{(1)}$  into  $a_2^{(2)}$  which is shown in figure (1.2). In this case  $\alpha'_0$  is identified with  $\alpha_0$  and  $\alpha'_1$  with  $\frac{1}{2}(\alpha_1 + \alpha_2)$ . In this case  $\alpha'_0$  is the longer root.

## 1.4 Affine Toda field theory with real coupling

### 1.4.1 The classical Mass Spectrum

Knowledge of the classical mass spectrum will prove useful in the discussion of the quantum theory later. \*

One way to find out about the classical mass spectrum is to expand the potential (1.9) for small  $\beta$

$$V(\Phi(x, t)) = \frac{m^2}{\beta^2} \sum_{i=0}^r n_i \left( \beta (\alpha_i \cdot \Phi(x, t)) + \frac{\beta^2}{2} (\alpha_i \cdot \Phi(x, t))^2 + \frac{\beta^3}{6} (\alpha_i \cdot \Phi(x, t))^3 \right) + O(\beta^2).$$

Due to the definition of the  $n_i$ 's the linear term vanishes. The quadratic and the cubic part can be rewritten defining the mass matrix and the three point coupling by

---

\* For non-selfdual theories the assumption will be that the fusing angles and mass ratios stay the same before and after quantisation.

$$\begin{aligned}
(\mathbf{M}^2)^{ab} &= m^2 \sum_{i=0}^r n_i \alpha_i^a \alpha_i^b \\
C^{abc} &= \beta m^2 \sum_{i=0}^r n_i \alpha_i^a \alpha_i^b \alpha_i^c.
\end{aligned}
\tag{1.15}$$

In terms of these the potential is

$$V(\Phi(x, t)) = \frac{1}{2} \Phi^a(x, t) (\mathbf{M}^2)^{ab} \Phi^b(x, t) + \frac{C^{abc}}{6} \Phi^a(x, t) \Phi^b(x, t) \Phi^c(x, t) + O(\beta^2).$$

For higher powers of the expansion  $n$ -point couplings can be derived in a similar fashion

$$C^{a_1 \dots a_n} = m^2 \sum_{i=0}^n n_i \alpha_i^{a_1} \dots \alpha_i^{a_n}.$$

A nice compilation of known facts about mass matrices and coupling constants can be found in [22]. Earlier results can be found in [16]. It is possible to determine masses and coupling constants explicitly given a suitable representation of the underlying algebra [22]. Because the couplings and masses will appear in the theory on a half line (Chapter 2) a summary for the  $a_n^{(1)}$  [22] and  $a_2^{(2)}$  theories will be given, following closely the presentation of the computation in [22].

#### 1.4.1.1 $a_n^{(1)}$

An exception to this theory is  $a_1^{(1)}$ , the sinh-Gordon theory as all three point couplings vanish for this model. This happens in no other theory. In the following  $n > 1$  will be assumed.\*

A task easier than in most other theories is the diagonalisation of the mass matrix. One has to find a particular representation for the simple roots. Define  $\omega = e^{\frac{2\pi i}{n+1}}$ , such that  $\omega^{n+1} = 1$ . This gives rise to a set of  $n$  complex  $n$ -dimensional vectors  $\gamma_j$  with components

$$\gamma_j^a = \omega^{aj}, \quad i = 0, \dots, n, \quad a = 1, \dots, n. \tag{1.16}$$

Since  $\sum_{k=0}^n \omega^k = 0$ ,

$$\gamma_i^* \gamma_j = (n+1) \delta_{ij} - 1.$$

---

\* A change to an imaginary coupling constant gives the sine-Gordon theory which has soliton solutions and a coupling dependent spectrum of breathers [58].

A complex representation of the roots of  $a_n^{(1)}$  is then given by

$$\alpha_i = \frac{1}{\sqrt{n+1}}(\gamma_{i+1} - \gamma_i)^*. \quad (1.17)$$

One can check that the usual relations  $\sum_{i=0}^n \alpha_i = 0$ , and

$$\alpha_i \cdot \alpha_j = \begin{cases} 0 & i \neq j \text{ and } i \neq j+1 \\ -1 & i = j+1 \\ 2 & i = j \end{cases}$$

are fulfilled. Choosing a complex basis for the scalar fields

$$(\Phi^a)^* = \Phi^{n+1-a}$$

one has the property

$$(\alpha_i^* \cdot \Phi)^* = \alpha_i^* \cdot \Phi. \quad (1.18)$$

This follows from (1.16) and (1.17). One consequence is that the potential in the Lagrangian is real as well

$$\mathcal{L} = \frac{1}{2} \partial \Phi^* \partial \Phi - \frac{m^2}{\beta^2} \sum_{i=0}^n e^{\beta \alpha_i^* \Phi}.$$

With the roots written as (1.16) the square of the mass matrix (1.18) is given by

$$\begin{aligned} (M^2)^{ab} &= \frac{m^2}{n+1} \sum_{i=0}^n (\gamma_{i+1}^* - \gamma_i^*)^a (\gamma_{i+1} - \gamma_i)^b \\ &= \begin{cases} 0 & \text{for } a \neq b \\ 4m^2 \sin^2 \frac{a\pi}{n+1} & \text{for } a = b \end{cases} \end{aligned}$$

So the matrix is obviously diagonal and the masses are

$$m_a = 2m \sin \frac{a\pi}{n+1}, \quad a = 1, \dots, n. \quad (1.19)$$

The masses for  $n = 2$  and  $n = 3$  are for example

$$\begin{aligned} a_2^{(1)} : \quad m_1 &= m_2 = \sqrt{3}m \\ a_3^{(1)} : \quad m_1 &= m_3 = \sqrt{2}m \text{ and } m_2 = 2m. \end{aligned}$$

When  $n$  is even every particle has a conjugate partner with the same mass. In the case  $n$  odd the heaviest particle is self-conjugate but the remaining particles again

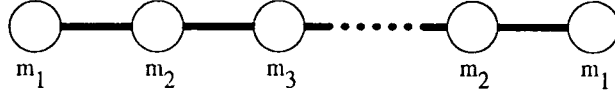


Fig. 1.3: The Dynkin diagram of  $a_n^{(1)}$  with masses associated to nodes.

occur as mass degenerate conjugate partners. This is related to the  $\mathbb{Z}_2$  symmetry of the  $a_n^{(1)}$  Dynkin diagram.

The relation (1.19) allows to associate the nodes of the Dynkin diagram unambiguously associated with the particles (see fig. (1.3)), the mass degeneration reflects the  $\mathbb{Z}_2$ -symmetry of the Dynkin diagram. The equation (1.19) also implies

$$m_{a-1} + m_{a+1} = 2m_a \cos \frac{\pi}{n+1} \quad a = 1, \dots, n$$

with  $m_0 = m_{n+1} = 0$ . Also, one should observe the following fact [22]. The  $n$ -component mass vector

$$\mathbf{m} = (m_1, \dots, m_n)$$

is an eigenvector of the Cartan matrix

$$C_{ab} = 2 \frac{\alpha_a \cdot \alpha_b}{\alpha_b^2}, \quad a, b = 1, \dots, r$$

of  $a_n^{(1)}$  with the eigenvalue  $4 \sin^2 \frac{\pi}{2(n+1)}$

$$C \mathbf{m} = 4 \sin^2 \frac{\pi}{2(n+1)} \mathbf{m}. \quad (1.20)$$

Thus the mass vector is the Perron-Frobenius eigenvector of the Cartan matrix, which guarantees that  $m_i > 0$ . So, for example for  $a_3^{(1)}$  the Cartan matrix and the mass vector are given by

$$C = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} \quad \text{and} \quad \mathbf{m} = \sqrt{2}m(1, \sqrt{2}, 1)$$

and clearly fulfill  $\mathbf{m}C = (2 - \sqrt{2})\mathbf{m}$ .

The three point coupling can be determined as well [22] using the same complex representation of the roots (1.17)

$$C^{abc} = \beta m^2 \sum_{i=0}^n \alpha_i^{*a} \alpha_i^{*b} \alpha_i^{*c}$$

$$= \begin{cases} 0 & a + b + c \neq 0 \bmod n + 1 \\ \frac{\beta m^2}{\sqrt{n+1}} (\omega^a - 1)(\omega^b - 1)(\omega^c - 1) & a + b + c = 0 \bmod n + 1 \end{cases}$$

Given that  $c = k(n + 1) - (a + b)$  for  $k = 1$  or  $2$  this simplifies for the non vanishing constants to the area rule

$$C^{abc} = \frac{\beta m^2}{\sqrt{n+1}} (\omega^a - 1)(\omega^b - 1)(\omega^c - 1) = \frac{i 2 \beta m_a m_b}{\sqrt{n+1}} \sin \frac{\pi(a+b)}{n+1}. \quad (1.21)$$

So the coupling is actually proportional to the area of a triangle with sides  $m_a$ ,  $m_b$  and  $m_c$ , where  $m_c$  is given by

$$m_c^2 = m_a^2 + m_b^2 + 2m_a m_b \cos \bar{\theta}_{ab}^c, \quad \bar{\theta}_{ab}^c = \pi - \theta_{ab}^c. \quad (1.22)$$

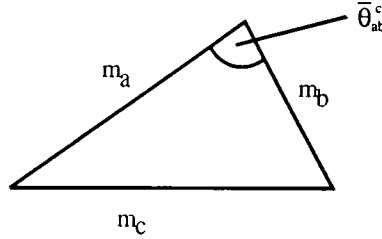


Fig. 1.4: The mass triangle

The angle  $\bar{\theta}_{ab}^c$  is the angle enclosed by  $m_a$  and  $m_b$ . The angle  $\theta_{ab}^c$  is the fusing angle for the reaction  $ab \rightarrow \bar{c}$ , a terminology that will become clear later. The allowed values of  $\theta_{ab}^c$  are

$$\theta_{ab}^c = \begin{cases} \frac{a+b}{n+1} \pi & \text{for } a + b + c = 1 + n \\ \left(2 - \frac{a+b}{n+1}\right) \pi & \text{for } a + b + c = 2(1 + n) \end{cases}$$

Note that the only dependence on the Lie algebra data in (1.21) is the factor  $\frac{1}{\sqrt{n+1}} = \frac{1}{h}$ , i.e the Coxeter number of  $a_n^{(1)}$ .

Before the end of the account on  $a_n^{(1)}$  some examples for the coupling constants and fusing angles for  $n = 2$  and  $n = 3$  are given:

- $a_2^{(1)}$ : There are only two non vanishing couplings

$$C^{111} = i2m_1^2\beta \sin \frac{2\pi}{3} = im_1^2\beta\sqrt{3} = -C^{222}, \quad \theta_{11}^1 = \frac{2\pi}{3}.$$

- $a_3^{(1)}$ : This theory has three non vanishing coupling constants

$$\begin{aligned} C^{112} &= \beta m_1 m_2 2i \sin \frac{\pi}{2} = i4m^2\beta, & \theta_{11}^2 &= \frac{\pi}{2} \\ C^{233} &= \beta 2im_2 m_3 \sin \frac{5\pi}{4} = -C^{112}, & \theta_{23}^3 &= -\frac{\pi}{2}. \end{aligned}$$

#### 1.4.1.2 $a_2^{(2)}$

As mentioned earlier in section (1.3.6) the algebra  $a_2^{(2)}$  is related to  $d_4^{(1)}$  and  $a_2^{(1)}$  by the folding procedure. Its mass matrix and coupling constant can be recovered from those two theories. For example the coupling constant of the single particle of  $a_2^{(2)}$  should be equal to  $C^{222}$  of  $d_4^{(1)}$ . But one can also straightforwardly expand the potential and read off the mass and coupling. The theory has been introduced earlier (1.14) and then the following values were chosen for the roots and Dynkin indices:  $\alpha_0 = -\frac{1}{\sqrt{2}}$ ,  $n_0 = 2$  and  $\alpha_1 = \sqrt{2}$ ,  $n_1 = 1$ . Using the definitions of the mass matrix and coupling constant (1.15) they are

$$\begin{aligned} m_1^2 &= 3m^2 \\ C^{111} &= \frac{3m^2\beta}{\sqrt{2}} = \frac{2\beta}{\sqrt{6}}m_1^2 \sin \frac{\pi}{3}. \end{aligned} \tag{1.23}$$

One sees that in this normalisation the only particle has the same mass as the  $a_2^{(1)}$  particles and also the same fusing angle (see p.17). The actual value of the coupling constant in terms of the particle mass is the same as that for the  $C^{222}$  coupling of  $d_4^{(1)}$  (see [22]).

### 1.4.1.3 Generalisations

A generalisation of equation (1.20) and an algebraic proof of it can be found in [27,28]. The generalised relation is

$$C Q_s = (2 - 2 \cos \frac{s\pi}{h}) Q_s$$

where  $s$  is the exponent of the underlying Lie-algebra and  $Q_s$  is a vector whose components are the conserved quantities  $Q_s^a$ .

The area formula (1.21) can be generalised to apply for any Lie algebra [34]

$$C_{ijk} = \lambda_{ijk} \frac{4\beta}{\sqrt{h^{(k)}}} m_a m_b \sin \bar{\theta}_{ab}^c.$$

The factor  $\lambda^{abc}$  takes care of the normalisation for different Lie algebras. The Coxeter number  $h$  has been replaced by  $h^{(k)}$  to ensure that for the Lie-algebra  $g^{(k)}$

$$h^{(k)} = k \cdot h$$

the fusing angles are an integer multiple of  $\frac{\pi}{h^{(k)}}$ . A general proof of the formula can be found in [27,28].

More details of the concepts mentioned in this section can be found in [22-25].

### 1.4.2 S-matrices

A classically integrable theory has infinitely many conserved quantities  $Q_s$ , where  $s$  is the spin. It is convenient to write the momentum  $p_a$  of a particle  $a$  of a two dimensional theory in terms of its rapidity  $\theta_a$  ( $v_a = \tanh \theta_a$ )

$$p_a = m_a (\cosh \theta_a, \sinh \theta_a).$$

In a quantum theory the conserved quantities of the classical theory correspond to locally conserved operators  $q_s^a$ . Single particle states are represented by simultaneous eigenstates of these operators. The locality of these operators implies that they act on multi-particle states additively. The evolution of multi particle states is described by the  $S$ -matrix of the quantum theory. Because the quantum theory has infinitely

many conserved operators the set of momenta and the number of particles for the in-going and the out-going multi-particle state is the same. Thus, there is no particle production. For the  $S$ -matrices this means a multi-particle  $S$ -matrix factorises into two-particle  $S$ -matrices. There are several arguments for this to happen, there is the wave argument by Zamolodchikov and Zamolodchikov [58], the wave packet argument by Witten and Shankar [59] and rigorous arguments were given by Iagolnitzer [60]. The  $S$ -matrix for an in-state of two particles  $A_i$  and  $A_j$   $|A_i(\theta_1)A_j(\theta_2)\rangle_{\text{in}}$  and an out-state  $|A_k(\theta_1)A_l(\theta_2)\rangle_{\text{out}}$  of particles  $A_k$  and  $A_l$  will be denoted as

$$|A_i(\theta_1)A_j(\theta_2)\rangle_{\text{in}} = S_{ij}^{kl}(\theta_{12})|A_k(\theta_1)A_l(\theta_2)\rangle_{\text{out}} \quad (1.24)$$

where the rapidity difference is written as  $\theta_1 - \theta_2 = \theta_{12}$ . Generally the  $S$ -matrix has to fulfil a cubic equation, the Yang-Baxter equation [58]

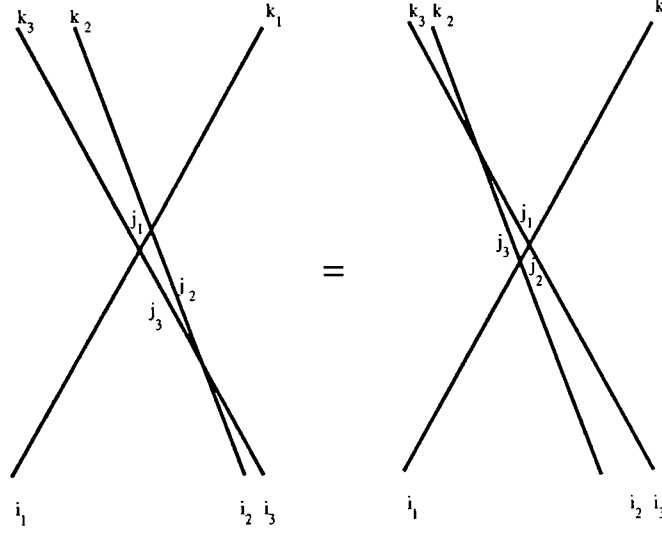


Fig. 1.5: Yang-Baxter equation

$$S_{j_2 j_3}^{k_3 k_2}(\theta_{23}) S_{j_1 j_3}^{k_1 j_3}(\theta_{13}) S_{i_1 i_2}^{j_1 j_2}(\theta_{12}) = S_{j_1 j_2}^{k_2 k_1}(\theta_{12}) S_{i_1 j_3}^{k_3 j_1}(\theta_{13}) S_{i_2 i_3}^{j_3 j_2}(\theta_{23}) \quad (1.25)$$

where a summation over all possible states allowed by the selection rules has to take place (fig. (1.5)). If the theory has no mass degenerate multiplets this equation is trivially fulfilled because the  $S$ -matrices are mere phases.



### 1.4.2.1 Affine Toda field theory with real coupling

Classically, for affine Toda field theory with real coupling there are no mass degenerate multiplets. Each particle is uniquely labelled by one of the higher spin charges. The spins of these charges are the exponents of the underlying Lie algebra modulo the Coxeter number  $h$  [61]. Now, it is tempting to assume that the quantum particle spectrum is essentially the same as the classical one. Also, the fusing angles of the classical and the quantum theory will be assumed to be the same. This assumption of the existence of infinitely many conserved quantities is substantiated by calculations by Niedermaier [62] of the first few of them in affine Toda field theory. However, for the non-simply laced algebras the assumption of the mass spectra to be exactly the same in the classical and the quantum case seems not to be correct[63], one has to make certain modifications. In the remaining section the main concern will be  $a_n^{(1)}$  and one should bear in mind that for other theories modifications may have to be made.

Lorentz invariance requires the action of the local operators on eigenstates to be

$$Q_s|p_a\rangle = q_s^a e^{s\theta}|p_a\rangle \quad s = p + kh \quad p, k \in \mathbb{Z}.$$

where the spin is given by the exponents of the underlying Lie-algebra. So, for example for  $a_n^{(1)}$  the spins are

$$s = 1, 2, 3, \dots, n \bmod (n+1).$$

As mentioned before it will be assumed that multi-particle states can be eigenstates of the local operators as well. For a two-particle state this means that

$$Q_p|p_a, p_b\rangle = (q_a^p e^{p\theta_a} + q_b^p e^{p\theta_b})|p_a, p_b\rangle.$$

As mentioned in the beginning there are no mass degenerate multiplets therefore the two-particle  $S$ -matrix is a mere phase for real  $\theta_{ab}$  and depends on the coupling constant  $\beta$  and the rapidity difference  $\theta_{ab}$  only. A two-particle scattering is described by

$$|p_a, p_b\rangle_{\text{in}} = S_{ab}(\theta_{ab})|p_a, p_b\rangle_{\text{out}}.$$

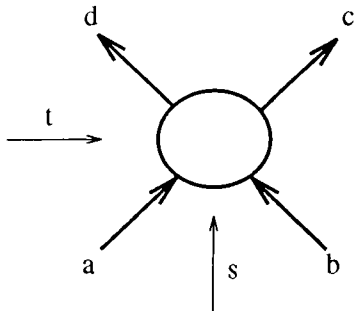


Fig. 1.6: Scattering of  $ab \rightarrow cd$  in the  $s$ -channel and  $a\bar{d} \rightarrow \bar{b}c$  in the  $t$ -channel

For a two-particle process there are two channels corresponding to the Mandelstam variables  $s$  and  $t$  to describe the scattering (see fig. (1.6)). The Mandelstam variable  $s$  is

$$s = (p_a + p_b)^2 = m_a^2 + m_b^2 + 2m_a m_b \cosh \theta_{ab}.$$

The rapidity difference can therefore be expressed in terms of the Mandelstam variable  $s$  [58]

$$\theta_{ab} = \cosh^{-1} \frac{s - m_a^2 - m_b^2}{2m_a m_b}.$$

Analytical continuation of  $s$  gives two square root branch cuts at  $(m_a \pm m_b)^2$ . In this notation the  $s$  channel corresponds to  $\text{Im}\theta_{ab} = 0$ ,  $\text{Re}\theta_{ab} > 0$  and the  $t$ -channel to  $\text{Im}\theta_{ab} = \pi$ ,  $\text{Re}\theta_{ab} < 0$ .

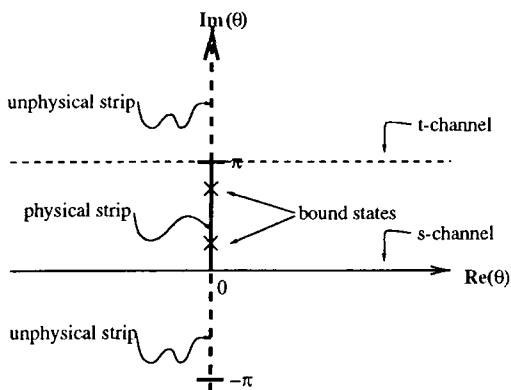


Fig. 1.7: The  $s$  and  $t$  channel and bound states in the complex  $\theta$  plane

Poles of the  $S$ -matrix corresponding to bound states lie in the physical strip,  $0 < \text{Im}\theta_{ab} < \pi$  (see fig. (1.7)). The  $S$ -matrix is a meromorphic function in  $\theta_{ab}$  and has to fulfil the following two requirements for all  $\theta_{ab}$

**Unitarity:** For any two particle process the probability resulting in a final state is one

$$S_{ab}(\theta_{ab})S_{ab}(-\theta_{ab}) = 1.$$

**Crossing:** For non self-conjugate particles  $a, b$  the transformation  $\theta_{ab} \rightarrow i\pi - \theta_{ab}$  implies the condition

$$S_{ab}(i\pi - \theta_{ab}) = S_{b\bar{a}}(\theta_{ab}).$$

This corresponds to saying the  $S$ -matrix is invariant under a change from the  $s$  to the  $t$  channel. A change from  $s$  to  $t$  corresponds to a change from  $\theta$  to  $i\pi - \theta$ . For self-conjugate particles the  $S$ -matrix is crossing symmetric.

These two conditions imply that the  $S$ -matrix is a  $2\pi i$  periodic function in the rapidity  $\theta_{ab}$ . This fact will be used to express it in terms of trigonometric functions [22].

A two-particle state may be dominated by a one-particle state  $|p_{\bar{c}}\rangle$  which is also part of the conjectured particle spectrum

$$Q_s|p_a, p_b\rangle \approx Q_s|p_{\bar{c}}\rangle.$$

Then the charges have to fulfil

$$q_a^p e^{p\theta_a} + q_b^p e^{p\theta_b} = q_{\bar{c}}^p e^{p\theta_{\bar{c}}} \quad (1.26)$$

This does not occur for any stable state  $c$  if the rapidity difference between the two particles is real,  $\theta_{ab} = \theta_a - \theta_b \in \mathbb{R}$ . The conserved charges for  $s = \pm 1$  correspond to the energy-momentum which has the consequence that (1.26) implies

$$m_{\bar{c}}^2 = m_a^2 + m_b^2 + 2m_a m_b \cosh \Theta_{ab} = m_a^2 + m_b^2 + 2m_a m_b \cos i\theta_{ab}^c. \quad (1.27)$$

This looks very similar to equation (1.22) encountered in the discussion of the classical mass spectrum which explains the expression fusing angle for  $\theta_{ab}^c$  introduced

there. Also, (1.27) implies that for imaginary  $\theta_{ab}^c$  the mass of  $m_c$  is smaller than that of  $m_a + m_b$ . Using the notation  $\bar{\theta}_{ab}^c = \pi - \theta_{ab}^c$  the rapidities  $\theta_a$  and  $\theta_b$  can be written as

$$\theta_a = \theta_{\bar{c}} - i\bar{\theta}_{ab}^c, \quad \theta_b = \theta_{\bar{c}} + i\bar{\theta}_{bc}^a.$$

If the vacuum state dominates an anti-particle state and the rapidity difference is  $\Theta_{a\bar{a}} = i\pi$  then it follows from (1.26) that

$$q_a^p = (-1)^{p+1} q_{\bar{a}}^p.$$

Therefore particles and anti particles are only distinguished by even spin charges. Also, theories with odd spins will only contain self-conjugate particles.

Since the Yang-Baxter equation (1.25) can not help to determine  $S$ -matrices one has to rely on the bootstrap principle to find consistency relations for the  $S$ -matrices. If the coupling constant of three particles  $a, b, c$  does not vanish,  $C^{abc} \neq 0$ , the particles  $a$  and  $b$  can fuse to form the bound state  $\bar{c}$ . There are two ways in which a fourth particle  $d$  can scatter with these three particles.

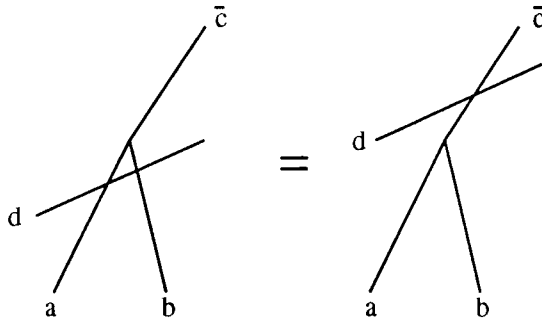


Fig. 1.8: Bootstrap principle

Due to the factorisation the  $S$ -matrix has to be the same in both cases (fig. (1.7))

$$S_{d\bar{c}}(\Theta) = S_{da}(\Theta - i\bar{\theta}_{ac}^b) S_{db}(\Theta + i\bar{\theta}_{bc}^a) \quad (1.28)$$

where  $\Theta = \theta_{\bar{c}} - \theta_d$  is the relative rapidity of  $\bar{c}$  and  $d$  [64]. For a two-particle state  $a$  and  $\bar{a}$  and a relative rapidity  $i\pi$  this agrees with the crossing relation. Using the

crossing relation the bootstrap equation can be written as

$$S_{da}(\Theta + i\theta_{ac}^b + i\theta_{bc}^a)S_{dc}(\Theta + i\theta_{bc}^a)S_{db}(\Theta) = 1$$

which is a product version of the charge bootstrap following from (1.27)

$$q_a^p e^{ip(\theta_{ac}^b + \theta_{bc}^a)} + q_c^p e^{ip\theta_{bc}^a} + q_b^p = 1.$$

The relation (1.28) and knowledge of classical couplings and fusion angles is enough to determine the  $S$ -matrix given a reasonable ansatz for all simply-laced theories.

#### 1.4.2.2 Explicit formulae for the $S$ -matrices

An element of the  $S$ -matrix ought to be unity for vanishing coupling  $\beta \rightarrow 0$ . For  $\beta \neq 0$  fixed poles indicating the fusing should be the only poles and they should be situated in the physical strip. This follows from the assumption that the classical mass spectrum is complete and the quantum theory has no new masses. So, the  $S$ -matrix has to contain some travelling zeros in the physical strip which cancel the fixed ones for  $\beta = 0$ . Unitarity requires each of the zeros to have an accompanying pole in the unphysical strip for non-vanishing coupling. Also, the  $S$ -matrix should exhibit a symmetry under  $\beta \rightarrow \frac{4\pi}{\beta}$  because it has been seen before (1.7) that the conformal Toda field theory has this symmetry after quantisation. It is useful to introduce a so-called block notation which allows one to write the  $S$ -matrices as factors

$$(x)_\theta = (x) = \sinh\left(\frac{\theta}{2} + \frac{i\pi x}{2h}\right) / \sinh\left(\frac{\theta}{2} - \frac{i\pi x}{2h}\right). \quad (1.29)$$

A crossing symmetric block is

$$[x] = (x)(h - x).$$

Some of the properties these blocks fulfil are

$$\begin{aligned} (0) &= 1, \quad (h) = -1, \quad (-x) = (x)^{-1}, \quad (x) = (x \pm 2h) \\ (x)_{\theta + \frac{i\pi x}{h}} (x)_{\theta - \frac{i\pi y}{h}} &= (x + y)_\theta (x - y)_\theta. \end{aligned} \quad (1.30)$$

A combination of fixed poles and travelling zeros may be written as

$$\{x\}_\theta = \{x\} = \frac{(x-1)(x+1)}{(x-1+B)(x+1-B)}. \quad (1.31)$$

In this equation  $B$  is a coupling dependent function

$$B = \frac{1}{2\pi} \frac{\beta^2}{1 + \frac{4\pi}{\beta^2}}. \quad (1.32)$$

Then  $\{x\}$  is unity for vanishing coupling and symmetric under  $\beta \rightarrow \frac{4\pi}{\beta}$  since  $B(\beta) = 2 - B(\frac{4\pi}{\beta})$ . Though in principle other functions for  $B(\beta)$  satisfying the constraints for (1.32) when  $\beta \rightarrow 0, \infty$  may be chosen, (1.32) is the usual choice which was first suggested by Arinshtein, Fateev and Zamolodchikov [15] for  $a_n^{(1)}$  by comparison with the sine/sinh-Gordon model [58].

The bootstrap allows one to find the  $S$ -matrices for the  $a_2^{(1)}$  theory as follows. The Coxeter number is 3 and the two particles in the theory are conjugate to each other  $1 = \bar{2}$ . The fusing angles (p. 19) are  $\theta_{11}^1 = \theta_{22}^2 = \frac{i2\pi}{3}$ . So,  $S_{11}$  needs a fixed pole at  $\frac{i2\pi}{3}$ , i.e. (2) describes the pole correctly. and a crossing symmetric  $S$ -matrix is given by

$$S_{11} = S_{22} = \{1\}.$$

Now the bootstrap (1.28) requires that

$$S_{12} = S_{1\bar{1}} = S_{11}(\Theta - i\frac{2\pi}{3})S_{11}(\Theta + i\frac{2\pi}{3})$$

which implies

$$S_{12} = \{2\}.$$

This expression has a zero at  $-\frac{i\pi}{3}$  and a pole at  $\frac{i\pi}{3}$ .

The  $S$ -matrices for all self-dual theories can be written down in closed form [22]. For example the  $S$ -matrices of the  $a_n^{(1)}$  theories can be written as

$$S_{ab} = \prod_{\substack{x=|a-b|+1, \\ \text{step } 2}}^{a+b-1} \{x\}.$$

Also, it is possible to express the  $S$ -matrices of the simply-laced series in terms of the root system of the underlying Lie-algebra [26]. For this one has to divide the indices of the roots of  $\alpha_n$  into two sets of mutually orthogonal roots

$$\bullet = \{1, \dots, k\} \text{ and } \circ = \{k+1, \dots, n\}.$$

The Weyl reflection  $w_i$  associated with a certain simple root  $\alpha_i$

$$w_i(x) = x - 2 \frac{x \cdot \alpha_i}{\alpha_i^2} \alpha_i$$

defines a Coxeter element  $w$

$$w = w_\bullet w_o = w_1 \dots w_k w_{k+1} \dots w_n$$

which is of order  $h$ , the Coxeter number of the algebra. A basis for the orbits of the Coxeter elements are the root vectors

$$\phi_i = w_n w_{n-1} \dots w_{i+1}(\alpha_i).$$

Then it is possible to write the  $S$ -matrix as

$$S_{ab} = \prod_{p=1}^n \left\{ 2p + 1 + \epsilon_{ab} \right\}_+^{\lambda_a \cdot \omega^{-p} \phi_b} \quad (1.33)$$

where  $\epsilon_{oo} = \epsilon_{\bullet\bullet} = 0$  and  $\epsilon_{o\bullet} = -\epsilon_{\bullet o} = 1$ . The "+" index means that because all blocks are accounted for by traversing the positive part of the orbit of  $\Phi_b$  only, for an extension of the product to the whole Coxeter orbit the numerator of the blocks are reconstructed by the positive part of the orbit and the denominator by the negative part.

To lower orders  $S$ -matrices have been checked by perturbation theory [25,29,30,65]. Also the quantum mechanical mass corrections of simply laced theories have been calculated and it was found that there is a universal renormalisation factor [29-31]. For the non-simply laced theories analytical investigations were done by Delius et al. [63]. Also, the mass ratios for one specific pair of non-self dual algebras has been tested numerically supplying evidence that the masses for these theories depend on the coupling constant as Delius suggested [66].

## 1.5 Affine Toda field theory with imaginary coupling

For affine Toda field theory with real coupling all solutions have been real and without singularities so far. This allowed one to interpret them as particles of the theory.

Also, the only real constant solution is  $\Phi = 0$  which is usually called the vacuum state.

If one allows complex solutions for the equations of motion (1.11) the picture changes. Instead of one constant solution there are infinitely many

$$\Phi = \frac{2\pi i}{\beta} \omega \quad \text{with } \omega \in \Lambda^*$$

where  $\Lambda^*$  is the co-root lattice. These solutions all have zero energy and all of them have the right to be called vacuum states. Usually the equations of motion of the theory allowing complex solutions are written with an imaginary coupling constant, i.e.  $\beta \rightarrow i\beta$  in (1.10) and (1.11) [35]

$$\mathcal{L} = \frac{1}{2} \partial_\mu \Phi^a(x, t) \partial^\mu \Phi^a(x, t) + \frac{m^2}{\beta^2} \sum_{i=1}^r e^{i\beta \alpha_i \cdot \Phi(x, t)} \quad (1.34)$$

and

$$\partial_\mu \partial^\mu \Phi(x, t) = -\frac{m^2}{i\beta} \sum_{i=0}^r n_i \alpha_i e^{i\beta \alpha_i \cdot \Phi(x, t)}. \quad (1.35)$$

Then the constant solution are

$$\Phi = \frac{2\pi}{\beta} \omega \quad \text{with } \omega \in \Lambda^*. \quad (1.36)$$

Soliton solutions are non-constant solutions which interpolate between these constant field configurations. It is instructive to study what happens to the sinh-Gordon equation (1.13) under a change in the coupling constant. Changing  $\beta \rightarrow i\beta$  in (1.13) one gets the sine-Gordon equation

$$\partial_\mu \partial^\mu \Phi(x, t) = -\frac{m^2}{\beta} 4 \sin \beta \Phi(x, t). \quad (1.37)$$

It is well known that the sine-Gordon equation has multi-soliton solutions. However, there is one significant difference between the solution of the sine-Gordon equation (1.37) and the equations of motion of the general case (1.35). The solution of (1.37) are real whereas the solutions to (1.35) are complex. But as Hollowood [35] has pointed out for the  $a_n^{(1)}$  theories, though the energy density of solutions to (1.35) is in general complex their energy-momentum is real. Using an algebraic method Olive et al. [45,67] were able to show that this true in the general case. This



algebraic approach is a generalisation of the Leznov-Saveliev solution [13] of the conformal Toda theory which has been discussed by Mansfield [68]. Hollowood [35] also mentioned that the masses of solitons and particles are proportional in the case of  $a_n^{(1)}$ . Later Olive et al. showed that this is true for other algebras as well [45]. Like masses of particles the masses of solitons can be associated to nodes of the Dynkin diagram. One important difference between solitons and particles is that a soliton with one specific mass may have several topological charges. This introduces a mass degeneracy of the theory which makes the quantisation, i.e. the finding of  $S$ -matrices, different from the approach taken for the real coupling theory because the Yang-Baxter equation (1.25) is not trivially fulfilled. A problem of the quantum theory is however that the complex solutions give rise to a classical Hamiltonian which is not positive definite. Consequently the quantum theory of these theories is non-unitary and the interpretation of the soliton solutions is unclear. Still, as pointed out in the beginning, theories with imaginary coupling are related to conformal field theories. So is, for instance, the theory associated with  $e_8$  and  $\beta^2 = -\frac{31}{32}$  corresponds to a conformal field theory with central charge  $c = \frac{1}{2}$ .

### 1.5.1 Hirota's Method

Soliton solutions can be found with Hirota's method [35-40,50,51]. Other approaches are Bäcklund transformations [70] and the algebraic approach [45,67]. The main idea of Hirota's method [69] is to change the variables of the equations of motion to an equation of "Hirota bi-linear type". To get an idea for an ansatz to achieve a correct change of variables one follows the lines of the calculations for Toda lattice equations (where  $\Phi$  depends on "t" only and not on "x"). The ansatz is

$$\Phi(x, t) = -\frac{1}{i\beta} \sum_{j=0}^r \eta_j \alpha_j \log \tau_j \quad (1.38)$$

where  $\eta_j = 2/\alpha_j^2$  [37]. Then the equations of motion (1.35) turn into the following expression

$$\sum_{j=0}^n \alpha_j P_j = 0.$$

Where the  $P_j$ 's are given as

$$2\tau_j^2 P_j = (\eta_j^2 (D_t^2 - D_x^2) \tau_j^2 - 2m^2 \eta_j \tau_j^2) \left( \prod_{k=0}^r \tau^{-\eta_k \alpha_k \cdot \alpha_j} - \mu_1 \right) = 0, \quad j = 0, \dots, r$$

where  $\mu_1$  is an arbitrary parameter. The operators  $D_x$  and  $D_t$  are Hirota's bilinear operators and are defined as

$$D_x^m D_t^n f(x, t)g(x, t) = \left( \frac{\partial}{\partial x} - \frac{\partial}{\partial x'} \right)^m \left( \frac{\partial}{\partial t} - \frac{\partial}{\partial t'} \right)^n f(x, t)g(x', t') \Big|_{x=x', t=t'}.$$

In general the coefficient  $P_j = \mu_2 n_j$  for some constant  $\mu_2$ . In order for the Hirota equation to hold for the lowest order of  $\epsilon$  one needs  $\mu_1 - \mu_2 = 0$  which implies  $P_j = 0$

$$\eta_j(\ddot{\tau}_j \tau_j - \dot{\tau}_j^2 - \tau_j'' \tau_j + \tau_j'^2) - 2m^2 n_j \tau_j^2 \left( \prod_{k=0}^r \tau^{-\eta_k \alpha_k \cdot \alpha_j} - 1 \right) = 0, \quad j = 0, \dots, r. \quad (1.39)$$

This formula holds for any affine Toda field theory. To find a solution one expands  $\tau_j$  in powers of an arbitrary parameter  $\epsilon$

$$\tau_j = 1 + \delta_j^{(1)} \exp(\phi) \cdot \epsilon + \delta_j^{(2)} \exp(2\phi) \cdot \epsilon^2 + \dots + \delta_j^{(p_j)} \exp(p_j \phi) \cdot \epsilon^{p_j}$$

where  $\phi = \sigma(x - vt) + \xi$  and  $\delta_j^{(k)}$ ,  $1 \leq k \leq p_j$  and  $\sigma, v$  and  $\xi$  are constant defining the shape, velocity and topological charge of the soliton. Introducing the eigenvalues  $\lambda$  of the matrix product  $NC$  with  $N = \text{diag}(N_0, n_1, \dots, n_r)$  and  $(C_{ij}) = \alpha_i \cdot \alpha_j$   $\sigma, v$  and  $m$  are related as

$$\sigma^2(1 - v^2) = m^2 \lambda.$$

Thus, there are different  $\tau$  functions for each  $\lambda$ . Furthermore  $p_j, n_j$  and  $\eta_j$  are related by

$$n_0 \eta_j p_j = n_j \eta_0 p_0.$$

This constitutes a generalisation of the work by Hollowood [35] and more details can be found in [36, 38].

## 1.5.2 Examples

### 1.5.2.1 $a_2^{(2)}$

As before the Lie algebra data is  $n_0 = 2, n_1 = 1, \alpha_0 = -\frac{1}{\sqrt{2}}, \alpha_1 = \sqrt{2}$ . In addition one needs  $\eta_0 = 4, \eta_1 = 1$  and  $p_0 = 1, p_1 = 2$  and finally  $\lambda = 3$ . The equations (1.39) yield the following two equations

$$\begin{aligned} (\ddot{\tau}_0 \tau_0 - \dot{\tau}_0^2 - \tau_0'' \tau_0 + \tau_0'^2) - m^2(\tau_1 - \tau_0^2) &= 0 \\ \ddot{\tau}_1 \tau_1 - \dot{\tau}_1^2 - \tau_1'' \tau_1 + \tau_1'^2 - 2m^2(\tau_0^4 - \tau_1^2) &= 0. \end{aligned}$$

These are solved by the  $\tau$  functions

$$\tau_0 = 1 \pm e^\phi \quad \text{and} \quad \tau_1 = 1 \mp 4e^\phi + e^{2\phi}$$

where  $\phi = \sqrt{3} \frac{m}{\sqrt{1-v^2}}(x - vt)$ . So, the solution can be written as

$$\begin{aligned} \Phi_\pm &= -\frac{1}{\beta} \sum_{i=0}^1 \eta_i \alpha_i \ln \tau_i \\ &= -\frac{\sqrt{2}}{\beta} \ln \frac{\tau_1}{\tau_0^2} = -\frac{\sqrt{2}}{\beta} \ln \frac{1 \mp 4e^\phi + e^{2\phi}}{(1 \pm e^\phi)^2}. \end{aligned} \quad (1.40)$$

This solution to (1.14) will be used in the next chapter for the static background solution of the  $a_2^{(2)}$  theory.

### 1.5.2.2 $a_n^{(1)}$

Assuming that (1.39) decouples one can rewrite it for the  $a_n^{(1)}$  case as

$$\ddot{\tau}_j \tau_j - \dot{\tau}_j^2 - \tau_j'' \tau_j + \tau_j'^2 = m^2(\tau_{j-1} \tau_{j+1} - \tau_j^2) \quad j = 0, 1, \dots, n,$$

The index  $j$  is modulo the Coxeter number  $h = n + 1$ . The one soliton solution can be written as

$$\tau_j^{(a)} = 1 + \exp[\Omega_a + \rho_a + ij\theta_a], \quad (1.41)$$

where,

$$\Omega_a = \sigma_a(x - u_a t), \quad \rho_a = \eta_a + i\xi_a, \quad \theta_a = \frac{2\pi a}{h}, \quad (1.42)$$

$\sigma_a, u_a, \eta_a, \xi_a \in \mathbb{R}$ . The parameter  $\sigma_a$  and the velocity  $u_a$  are related by

$$\sigma_a^2(1 - u_a^2) = 4m^2 \sin^2 \frac{\pi a}{h}. \quad (1.43)$$

There are  $n$  species of single soliton solutions with a certain number of topological charges each for an affine Toda field theory corresponding to  $a_n^{(1)}$ . The superscript of  $\tau_j^{(a)}$  indicates to which species it corresponds. Sometimes it is useful to write  $\tau$ -functions in terms of “light-cone”-coordinates,  $x_\pm = \frac{1}{\sqrt{2}}(t \pm x)$ , the quantity  $\Omega_a$  can be written as

$$\Omega_a = \delta_a^{(-)} x_+ - \delta_a^{(+)} x_-,$$

with,

$$\delta_a^{(\pm)} = \frac{1}{\sqrt{2}}\sigma_a(1 \pm u_a).$$

As well as one soliton solutions multi-soliton solutions can be achieved with Hirota's method. An  $N$  soliton solution is obtained by setting  $t_j^{(a)} = 0$  in the expansion of  $\tau_j$  for  $N < a$

$$\tau_j = \sum_{a=0}^N \epsilon^a t_j^{(a)}.$$

The two-soliton solution may be written as

$$\tau_j^{(ab)} = 1 + \exp[\Omega_a + \rho_a + ij\theta_a] + \exp[\Omega_b + \rho_b + ij\theta_b] + A_{(ab)} \exp[\Omega_+ + \rho_+ + ij\theta_+] \quad (1.44)$$

where  $\Omega_+ = \Omega_a + \Omega_b$ ,  $\rho_+ = \rho_a + \rho_b$  and  $\theta_+ = \theta_a + \theta_b$ , which can be compactly written as,

$$\tau_j^{(ab)} = 1 + (\tau_j^{(a)} - 1) + (\tau_j^{(b)} - 1) + A_{(ab)}(\tau_j^{(a)} - 1)(\tau_j^{(b)} - 1).$$

$A_{(ab)}$  is the interaction coefficient of the two solitons  $\tau_j^{(a)}$  and  $\tau_j^{(b)}$ . This phrase suggests that the two-soliton solution  $\tau_j^{(ab)}$  can be thought of being constructed from the two single-soliton solutions [35]. The interaction coefficient is given by

$$A_{(ab)} = -\frac{(\sigma_a - \sigma_b)^2 - (\sigma_a u_a - \sigma_b u_b)^2 - 4m^2 \sin^2(\frac{\pi}{h}(a-b))}{(\sigma_a + \sigma_b)^2 - (\sigma_a u_a + \sigma_b u_b)^2 - 4m^2 \sin^2(\frac{\pi}{h}(a+b))} \quad (1.45a)$$

$$= \frac{\sin\left(\frac{\Theta}{2i} + \frac{\pi(a-b)}{2h}\right) \sin\left(\frac{\Theta}{2i} - \frac{\pi(a-b)}{2h}\right)}{\sin\left(\frac{\Theta}{2i} + \frac{\pi(a+b)}{2h}\right) \sin\left(\frac{\Theta}{2i} - \frac{\pi(a+b)}{2h}\right)}. \quad (1.45b)$$

In (1.45b) the rapidity difference  $\Theta = \Theta_a - \Theta_b$ , with  $u_a = \tanh \Theta_a$  was introduced. For details of the construction of soliton solutions of more than two solitons see [35]. It is worth noticing that the general  $N$ -soliton solution depends on two-soliton interaction coefficients only. For example the  $\tau$ -function of the three-soliton solution is given by

$$\begin{aligned} \tau_j = & 1 + e^{\Phi_j^{(a)}} + e^{\Phi_j^{(b)}} + e^{\Phi_j^{(c)}} + A_{ab}e^{\Phi_j^{(a)}}e^{\Phi_j^{(b)}} + A_{cb}e^{\Phi_j^{(c)}}e^{\Phi_j^{(b)}} + A_{ac}e^{\Phi_j^{(a)}}e^{\Phi_j^{(c)}} \\ & + A_{ab}A_{bc}A_{ac}e^{\Phi_j^{(a)}}e^{\Phi_j^{(b)}}e^{\Phi_j^{(c)}}, \end{aligned}$$

where  $A_{ab}$  is the interaction coefficient in (1.45) and  $\Phi_j^{(k)} = \Omega_k + \rho_k + ij\theta_k$ . This is somewhat reminiscent of the situation for the  $S$ -matrix of affine Toda field theory with real coupling which factories into two-particle  $S$ -matrices.

### 1.5.3 Energy

As mentioned before, the energy-momentum tensor of soliton solutions is complex, but the energy is still real. An elegant method to see this for single and multi soliton solutions was described in [45]. Following this paper the energy-momentum tensor can be written as

$$T_{\mu\nu} = (\eta_{\mu\nu}\partial^2 - \partial_\mu\partial_\nu)C. \quad (1.46)$$

Alternatively (1.46) can be expressed in light-cone components as

$$T_{+-} = \partial_+\partial_-C, \quad (1.47a)$$

$$T_{\pm\pm} = -\partial_\pm^2C. \quad (1.47b)$$

$T_{+-}$  is the trace of the energy-momentum tensor (1.46) and (1.47) can be written as

$$\partial^2C = -\frac{2m^2}{\beta^2} \sum_{j=0}^n (e^{i\beta\alpha_j\phi} - 1).$$

The function  $C$  can be determined up to a constant using the one-soliton solution

$$C = -\frac{2}{\beta^2} \sum_{j=0}^n \ln \tau_j.$$

Because the mass  $\mathcal{E}$  and the momentum  $\mathcal{P}$  densities are given by components of the energy-momentum tensor,  $\mathcal{E} = T_{00}$  and  $\mathcal{P} = T_{10}$ , the mass of the soliton can be calculated as follows. Consider the light-cone energy-momentum density

$$\mathcal{P}^\pm = \frac{\mathcal{E} \pm \mathcal{P}}{\sqrt{2}},$$

which tells one with the help of (1.46) that

$$P^+ = (-\partial_+C)_{x=-\infty}^\infty, \quad P^- = (\partial_-C)_{x=-\infty}^\infty.$$

To evaluate the limits one has to look at the single soliton solution for which

$$\partial_\pm C = \mp \frac{2}{\beta^2} \sum_{j=0}^n \frac{\delta_a^\mp (\tau_j^{(a)} - 1)}{\tau_j^{(a)}}.$$

The ratios  $\frac{(\tau_j^{(a)} - 1)}{\tau_j^{(a)}}$  vanish in the limit  $x \rightarrow -\infty$  and tend to 1 for  $x \rightarrow \infty$ . Using (1.43) and writing the rapidity of the soliton as  $\Theta_a = \frac{1}{2} \ln \left( \frac{1+u_a}{1-u_a} \right)$  allows to write the energy-momentum tensor of a single soliton as

$$P^\pm = \frac{4hm}{\sqrt{2}\beta^2} \sin\left(\frac{\theta_a}{2}\right) e^{\mp\Theta_a}.$$

Therefore its mass is

$$M_a^2 = 2P^+P^- = \left( \frac{4hm}{\beta^2} \sin\left(\frac{\theta_a}{2}\right) \right)^2.$$

Apparently the mass of a species  $a$  soliton is proportional to the mass of the fundamental Toda particle of  $a_n^{(1)}$  affine Toda field theory,  $m_a = 2m \sin(\frac{\theta_a}{2})$ , as mentioned before.

A similar calculation for multi-soliton solutions can be done and in the special case of two solitons of species  $a$  and  $b$  the result for the energy-momentum tensor is

$$\sqrt{2}P^\pm = M_a e^{\mp\Theta_a} + M_b e^{\mp\Theta_b}. \quad (1.48)$$

In [35,37] it was shown that solitons of the species  $a$  carry topological charges which lie in the highest weight representation of the  $a^{th}$  fundamental weight of the  $a_n$  algebra. Therefore it is natural to associate the species of the soliton with the nodes of the Dynkin diagram of the associated Lie algebra  $a_n$ .

#### 1.5.4 Topological Charges

In the soliton solutions of the complex affine Toda field equations, the topological charge is a conserved quantity of zero spin. For the  $a_n^{(1)}$  series, topological charges of the single and multi-soliton solutions have been calculated [37] .

The topological charge  $q$  of a solution  $\phi$  is defined by

$$q = \frac{\beta}{2\pi} \int_{-\infty}^{\infty} \partial_x \phi dx = \frac{\beta}{2\pi} \lim_{x \rightarrow \infty} (\phi(x, t) - \phi(-x, t)). \quad (1.49)$$

Writing the solution  $\phi$  in terms of the breather  $\tau$ -functions, defining  $f_j = \frac{\tau_j}{\tau_0}$  for  $j = 1 \dots n$  and making use of the definition of the logarithm of a complex number,

(1.49) can be recast as

$$q = -\frac{1}{2\pi i} \sum_{j=1}^n \alpha_j \lim_{x \rightarrow \infty} \left[ \ln |f_j(x, t)| - \ln |f_j(-x, t)| + i \arg(f_j(x, t)) + 2i\pi k' - i \arg(f_j(-x, t)) - 2i\pi k'' \right], \quad (1.50)$$

where  $k', k'' \in \mathbb{Q}$ . A simplification of (1.50) results from the fact that

$$\lim_{|x| \rightarrow \infty} |f_j(x, t)| = 1,$$

thus

$$q = -\frac{1}{2\pi} \sum_{j=1}^n \alpha_j \lim_{x \rightarrow \infty} \left[ \arg(f_j(x, t)) - \arg(f_j(-x, t)) + 2\pi k \right] \quad (1.51)$$

with  $k = k' - k''$ . The number  $k$  determines the curve  $f_j$  in the complex plane, and in particular, how often and in what direction it winds around the origin. The topological charge is therefore determined by the change in the argument of  $f_j$  as  $|x|$  goes to infinity.

For the  $a_n^{(1)}$  theory the following facts are known concerning topological charges [37]. For the  $a^{\text{th}}$  single soliton, i.e. the one corresponding to the  $a$ -th node of the Dynkin diagram, there are

$$\tilde{h}_a = \frac{h}{\gcd(a, h)}$$

different topological charges. The “highest topological charge” is

$$t_a^{(1)} = \sum_{j=0}^n \frac{a(h-j) \bmod h}{h} \alpha_j.$$

The general formula for all  $\tilde{h}_a$  topological charges is

$$t_a^{(k)} = \sum_{j=1}^n \frac{a(h-j) \bmod h}{h} \alpha_j - \sum_{l=1}^{k-1} \sum_{j=1}^n \delta_{a(h-j) \bmod h, h-l \gcd(a, h)} \alpha_j.$$

Furthermore it can be shown that the topological charges of the  $a$ -th single soliton lie in the  $a$ -th fundamental representation. One problem is that the topological charges fill the fundamental representations for the end nodes only. The construction of topological charges of multi solitons is relatively straightforward and it turns out that their topological charges is the sum of the topological charges of their constituent solitons.

## Chapter II

### Boundaries in Affine Toda Field Theory

#### 2.1 Field Theories on a half Line

Studies of two dimensional field theories have been focussed on those defined on the full line,  $x_1 = x \in \mathbb{R}$ ,  $x_0 = t \in \mathbb{R}$ . These theories are often called bulk theories. Since many physical systems are finite in their spatial dimensions it is interesting to study theories which are defined for a finite line or at least for a half line, e.g.  $x \leq 0$ . Such a theory should take boundary effects into account. In the past few years there has been progress in the understanding of integrable systems defined on the half-line. The foundations of this work were laid by Cherednik [71] more than a decade ago. He formulated an algebraic approach to scattering on a half line,  $x \leq 0$ . In terms of field theory this can be stated as follows. Let  $D$  be a dynamical system integrable on the full-line which implies the factorisability of the  $S$ -matrix then the assumption are

- a) When  $D$  is restricted to the half-line the particle content (mass spectrum) does not change;
- b) The  $S$ -matrices describing the mutual interactions of particles are not changed,
- c) The boundary reflects particles elastically (up to rearrangements of mass degenerate particles).

##### 2.1.1 Classical and Quantum Integrability

One of the first theories to receive some attention was the sine-Gordon theory. As with all other theories there were two main steps. Firstly one had to find out under which condition the classical theory would be integrable on a half line. Secondly one had to determine the reflection matrices which describe the influence of the boundary.



For the Sine-Gordon model both problems were solved by Ghoshal and Zamolodchikov [72,73]. They conjectured that the most general boundary potential leaving the classical theory integrable was

$$\mathcal{B}(\Phi) = M \cos \beta \left( \frac{\Phi - \Phi_0}{2} \right) \quad (2.1)$$

where  $M$  and  $\Phi_0$  are arbitrary constants and  $\beta$  is the coupling constant for the Lagrangian

$$\mathcal{L}_0 = \frac{1}{2}(\partial_\mu \Phi)^2 - \frac{m^2}{\beta^2} \cos(\beta \Phi).$$

The entire Lagrangian for the theory on the half line can then be written as

$$\tilde{\mathcal{L}} = \theta(-x)\mathcal{L}_0 - \delta(x)\mathcal{B}. \quad (2.2)$$

This formula for  $\Phi_0 = 0$  or  $\Phi_0 = \frac{\pi}{2}$  has appeared in classical considerations earlier [74-76]. Later McIntyre, and Saleur et al. gave a proof of this conjecture independently [77,78]. Ghoshal and Zamolodchikov found their conjecture by investigating under which conditions the first non-trivial integral of motion stays conserved for a theory with boundary. Generally, one would expect the boundary to harm some of the infinitely many conserved quantities. The introduction of a boundary destroys, for example, the translational invariance which means the momentum is no longer conserved. Also, the topological charge of the soliton solutions is conserved only for the case  $M \rightarrow \infty$  [72,73].

For the second problem, the scattering theory, new relations which generalise the Yang-Baxter equation, crossing unitarity and the bootstrap principle have to be found. Following Cherednik's rules outlined above the presence of a boundary is described by the introduction of reflection matrices  $K_a(\theta_a)$  for a particle  $a$  with rapidity  $\theta_a$  ( $v_a = \tanh \theta_a$ )

$$|a, \theta_a \rangle_{\text{out}} = K_a(\theta_a) |a, -\theta_a \rangle_{\text{in}}. \quad (2.3)$$

In general,  $K_a(\theta)$  should mix two particles states  $a$  and  $b$ , but for affine Toda field theory there will only be one reflection matrix for each particle. For real  $\theta$ ,  $K$  has to fulfil a unitarity condition just like the  $S$ -matrix

$$K_a(\theta_a) K_{\bar{a}}(-\theta_a) = 1. \quad (2.4)$$

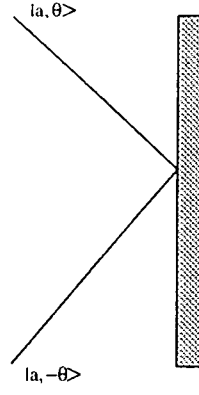


Fig. 2.1: Reflection of a particle on a boundary

Cherednik [71] found the generalisation of the Yang-Baxter equation, the reflection Yang-Baxter equation

$$K_2(\theta_2)S(\theta_1 + \theta_2)K_1(\theta_1)S(\theta_1 - \theta_2) = S(\theta_1 - \theta_2)K_1(\theta_1)S(\theta_1 + \theta_2)K_2(\theta_2). \quad (2.5)$$

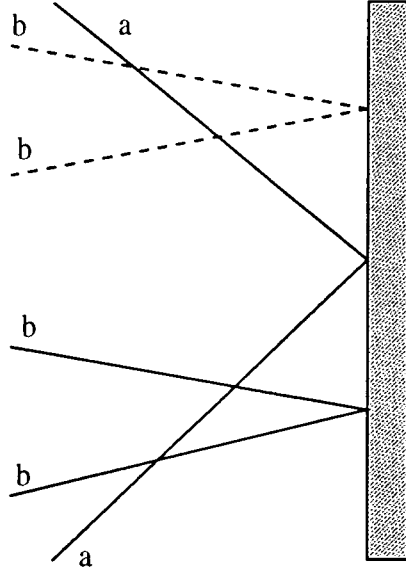


Fig. 2.2: The boundary Yang-Baxter equation

The Yang-Baxter equation can be understood and analysed with the help of quantum groups. Sasaki mentions that the reflection Yang-Baxter equation can be treated in terms of the reflection equation algebra which is related and inherits a lot properties from quantum groups (references in [79]). Equation (2.5) has no consequences for affine Toda field theory because the  $K$  and  $S$ -matrices of this theory are diagonal. More important is the crossing unitarity condition found by Ghoshal and Zamolodchikov [72,73]

$$K_a(\theta)K_{\bar{a}}(\theta - i\pi) = S_{aa}(2\theta) = S_{\bar{a}\bar{a}}(2\theta) \quad (2.6)$$

and the reflection bootstrap equation found by Fring and Köberle [80,81]

$$K_c(\theta) = K_a(\theta + i\bar{\theta}_{ac}^b)K_b(\theta + i\bar{\theta}_{bc}^a)S_{ab}(2\theta + i\bar{\theta}_{ac}^b - i\bar{\theta}_{bc}^a), \quad (2.7)$$

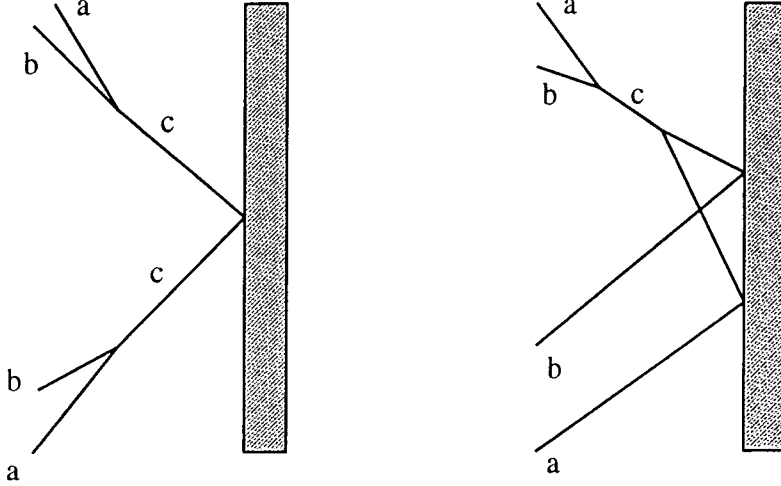


Fig. 2.3: The reflection bootstrap

where the angles  $\bar{\theta}_{ab}^c$  are the fusing angles discussed in the context of  $S$ -matrices earlier. If the reflection boundary equation applies (an exception is for instance  $a_1^{(1)}$ ) the crossing condition (2.6) is a consequence of (2.7) when all fusings are taken into account as pointed out by Sasaki [79]. But solutions of (2.6) do not necessarily solve (2.7). Also, Sasaki points out that it is easy to generate more solutions to the bootstrap equations once a single solution is known. Compare for instance the bootstrap equation for the  $S$ -matrix and the one for the reflection matrix

$$S_{cd} = S_{ad}(\theta + i\bar{\theta}_{ac}^b)S_{bd}(\theta - i\bar{\theta}_{bc}^a)$$

$$K_c(\theta) = K_a(\theta + i\bar{\theta}_{ac}^b)K_b(\theta - i\bar{\theta}_{bc}^a)S_{ab}(2\theta + i\bar{\theta}_{ac}^b - i\bar{\theta}_{bc}^a).$$

Assuming  $K_c(\theta)$  is a solution it follows that

$$K_c'(\theta) = K_c(\theta)(S_{cd}(\theta))^{\pm 1} \quad (2.8)$$

is a solution as well for arbitrary  $d$ . Assuming the CPT invariance of the  $S$ -matrix survives in the half-line theory there is another possible solution

$$K_a(\theta) = K_{\bar{a}}(\theta).$$

Similarly

$$K'_a(\theta) = K_a(\theta - i\pi) = S_{aa}(2\theta)/K_a(\theta). \quad (2.9)$$

follows from the crossing condition (2.6). Finally Sasaki mentions the reflection matrix chosen as

$$K_a(\theta) = \sqrt{S_{aa}(2\theta)}$$

formally satisfies the reflection bootstrap equation. But Sasaki does not make further use of the equations ((2.8) and following) and considers meromorphic solutions only.

## 2.2 Reflection matrices for affine Toda field theory

Assuming the integrability of the quantum theory and the known exact  $S$ -matrices, Fring and Köberle [80,81] and Sasaki have worked out reflection matrices for the *ade* series of affine Toda field theory with real coupling. Sasaki gives examples for some algebras whereas Fring and Köberle give general formulae and in the later publication they give examples for some non-simply laced algebras. Sasaki introduces a new block notation for the reflection matrices similar to the one for  $S$ -matrices. Using  $(x)$  defined in (1.29) define  $[x]$  as

$$[x] = [x]_\theta = \frac{(x - \frac{1}{2})(x + \frac{1}{2})}{(x - \frac{1}{2} + \frac{B}{2})(x + \frac{1}{2} - \frac{B}{2})}.$$

The  $S$ -matrix block at  $2\theta$  can be expressed as

$$\{x\}_{2\theta} = \frac{[x/2]_\theta}{[h - x/2]_\theta} = \frac{[x/2]}{[h - x/2]}.$$

Now it is worthwhile to look briefly at two of the examples given by Sasaki.

$a_1^{(1)}$  : As mentioned earlier the three point coupling for the only, neutral, particle of this theory vanishes which means that the reflection bootstrap equation (2.7) is void. But the crossing unitarity equation (2.6) gives rise to the following reflection matrices assuming a minimality of the number of poles

$$K_1(\theta) = [1/2] \text{ or } [3/2]^{-1} \text{ for } S_{11} = [1/2][3/2].$$

$a_2^{(1)}$  : This theory is the first in the  $\overline{a_n^{(1)}}$  series where the reflection bootstrap will be used. The theory has two particles 1, 2 with  $1 = \bar{2}$  and  $h = 3$ . The reflection bootstrap equation (2.7) are

$$\begin{aligned} K_2(\theta) &= K_1(\theta + i\frac{\pi}{3})K_1(\theta - i\frac{\pi}{3})S_{11}(2\theta) \\ K_1(\theta) &= K_2(\theta + i\frac{\pi}{3})K_2(\theta - i\frac{\pi}{3})S_{11}(2\theta) \end{aligned}$$

The  $S$ -matrix of this theory is  $S_{11}(2\theta) = [1/2]/[5/2]$ . One obvious solution with two poles and zeros is

$$K_1 = 1, K_2(\theta) = S_{11}(2\theta) \text{ or } K_1(\theta) = S_{11}(2\theta), K_2 = 1.$$

Another solution with six poles and zeros is

$$K_1 = K_2 = [1/2][3/2] \text{ or } [3/2][5/2]^{-1}$$

which are related by (2.9).

Sasaki gives far more examples but instead of listing all of them one might rather compare the  $a_2^{(1)}$  result with that of Fring and Köberle [80,81]. They generate their solutions from the reflection bootstrap equation (2.7) as well. In their notation they are looking for a wall matrix  $W_i$  which is  $K_i$  in the above notation. Amongst others they give a formula for reflection matrices of the  $a_n^{(1)}$  series

$$W_i(\theta) = \prod_{l=1}^{\mu(i)} \mathcal{W}_{n+2\nu(l)-2\mu(i)}(\theta). \quad (2.10)$$

Here some new notation has to be introduced

$$\mu(i) = \begin{cases} i & i \leq [h/2] \\ h-i & i > [h/2] \end{cases}, \quad \nu(i) = \begin{cases} i & i \text{ odd} \\ i+h & i \text{ even} \end{cases}$$

$$\mathcal{W}_x(\theta) = \frac{w_{1-x}(\theta)w_{-1-x}(\theta)}{w_{1-x-B}(\theta)w_{-1-x+B}(\theta)} = [-x/2 \pm h/2]$$

and finally

$$w_x(\theta) = -\left(\frac{x \pm h}{2}\right).$$

For the case  $a_2^{(1)}$  one has the following data  $\mu(1) = \mu(2) = 1$  and the resulting matrices due to the  $\pm$  sign are

$$W_1 = W_2 = [-5/2] \text{ or } [1/2].$$

These two factors do not obey the reflection bootstrap (2.7). Only their coupling independent factors satisfy this equation. Fring and Köberle call this the minimal theory. For  $a_3^{(1)}$  the reflection coefficients coincide with those of Sasaki. So, the formula can be expected to work for  $a_{\text{even}}^{(1)}$  cases only and has not been proven in the general case. Recently, Kim [82] has conjectured a general formula for the reflection matrices he calculated perturbatively in terms of the root system similar to the notation which can be used to write the  $S$ -matrix (1.33). He derives a matrix  $J_a(\theta)$  which is essentially the reflection matrix

$$J_a(\theta) = K_a(\theta)/\sqrt{S_{aa}(2\theta)}.$$

With the definition of  $\epsilon_\bullet = 1$  and  $\epsilon_o = 0$  the matrix  $J_a(\theta)$  is given by

$$J_b(\theta) = \prod_{p=0}^{h-1} [2p + \frac{1}{2} + \epsilon_b]^{\frac{1}{2}(\sum_a \lambda_a w^{-p} \phi_b)}. \quad (2.11)$$

The factor  $\frac{1}{2}$  is probably given because the sum is over all orbits not just the positive ones as in (1.33). Since both (2.10) and (2.11) are products it might be interesting to investigate whether (2.11) is in some way the “correct” version of (2.10).

### 2.2.1 Missing Link

So far only the solutions to the reflection bootstrap equation correspond in the classical limit to only one type of boundary condition, the natural Neumann condition [83,84]. But affine Toda field theories allow classically non-trivial boundary conditions (see next section). It is unclear how their reflection coefficients for non-trivial boundary conditions of the classical theories correspond to reflection matrices in the quantum theory. Also, there is no way to say why one should prefer one solution to the reflection bootstrap equation to the other.

In the weak coupling limit, the classical regime,  $\beta \rightarrow 0$ , the  $S$ -matrix becomes the unit matrix and (2.7) transforms to the classical reflection bootstrap equation

$$K(\theta_c) = K(\theta_a)K(\theta_b) \quad (2.12)$$

where  $\theta_a = \theta_c - i\bar{\theta}_{ac}^b$ ,  $\theta_b = \theta_c + i\bar{\theta}_{bc}^a$  ( $\bar{\theta} = \pi - \theta$ ). The rapidities correspond to particle fusions  $ab \rightarrow c$ , i.e  $\theta_c$  has to be a pole of the  $S$ -matrix. In the context of integrable boundary conditions for the classical affine Toda field theory [85-88] it was discovered that solutions to (2.12) can be found from reflection factors. This will be discussed in the following section. From the solutions of (2.12) one can construct solution of (2.7) which have a dependence on the boundary potential.

For the affine Toda field theory with natural Neumann boundary conditions reflection matrices have been worked out perturbatively by Kim [83,84]. Again some of these coincide with the ones found by Sasaki. As already mentioned, recently it turned out that these solutions can be written in terms of the root system (2.11) similar to  $S$ -matrices (1.33) [82].

## 2.3 Boundary conditions for affine Toda field theory

Integrable boundary conditions for the  $a_n^{(1)}$  series affine Toda field theory have been conjectured first in [85]. In a later publication [86] this conjecture was generalised to all simply laced theories. The Lagrangian  $\bar{\mathcal{L}}$  of affine Toda theory on the half line differs from affine Toda theory (1.10) on the full line. The Lagrangian for the half line theory is given by (2.2)

$$\bar{\mathcal{L}} = \theta(-x)\mathcal{L} - \delta(x)\mathcal{B}.$$

The boundary potential  $\mathcal{B}$  depends on the fields  $\phi(x, t)$  but not their derivatives. Therefore the equations of motion are (1.11)

$$\partial_\mu \partial^\mu \phi(x, t) = -\frac{m^2}{\beta} \sum_{i=0}^r n_i \alpha_i e^{\beta \alpha_i \cdot \phi(x, t)}$$

with the restriction  $x \leq 0$  and

$$\left. \frac{\partial \phi}{\partial x} \right|_{x=0} = -\frac{\partial \mathcal{B}}{\partial \phi} = -\frac{m}{2\beta} \sum_{i=0}^r \alpha_i A_i e^{\frac{\beta}{2} \alpha_i \cdot \phi}. \quad (2.13)$$

The generic form for  $\mathcal{B}$  is

$$\mathcal{B} = \frac{m}{\beta^2} \sum_{i=0}^r A_i e^{\frac{\beta}{2} \alpha_i \cdot \phi} \quad (2.14)$$

with  $A_i \in \mathbb{R}$ , which generalises (2.1). Except for  $a_1^{(1)}$  the coefficients  $A_i$  are constrained for all non-simply laced algebras by

$$|A_i| = \begin{cases} 2\sqrt{n_i} & \text{for } i = 1 \dots n \\ 0 & \text{for } i = 1 \dots n \text{ (natural Neumann condition).} \end{cases} \quad (2.15)$$

The sinh-Gordon theory is an exception which allows an continuous deformation of any boundary condition to those of the natural Neumann conditions. This theory will be discussed in some detail later. The conjecture (2.15) was subsequently proved [87] by finding a Lax-pair representation of the boundary problem. Also, it was shown that a more general boundary condition that includes time derivatives leads to even stricter conditions on the boundary potential [88]. One should note here that the  $a_n^{(1)}$  theory is special in the sense that it allows the solution  $\phi = 0$  for non-trivial symmetric boundary conditions. Other theories only allow this for the natural Neumann condition.

### 2.3.1 Evidence and Proof of classical Integrability

Initially the above result for the boundary potential was found for  $a_n^{(1)}$  by looking at spin  $\pm 2$  charges of the half-line theory [85]. Similar calculations had been done for the full line before [62,63]. A general formula for the spin  $\pm 3$  densities on the whole line using light-cone coordinates ( $x^\pm = (x^0 \pm x^1)/\sqrt{2}$ ) is

$$T_{\pm 3} = \frac{1}{3} A_{abc} \partial_\pm \phi_a \partial_\pm \phi_b \partial_\pm \phi_c + B_{ab} \partial_\pm^2 \phi_a \partial_\pm \phi_b$$

where the coefficients  $A_{abc}$  are completely symmetric and the coefficients  $B_{ab}$  completely anti-symmetric. The spin  $\pm 3$  density corresponds to a spin  $\pm 2$  charge. To construct conserved quantities the densities have to satisfy

$$\partial_\pm T_{\pm 3} = \partial_\pm \Theta_{\pm 1}$$

where  $\Theta_{\pm 1}$  can be calculated to be

$$\Theta_{\pm 1} = -\frac{1}{2} B_{ab} \partial_\pm \phi_a \frac{\partial V}{\partial \phi_b}$$

with the constraint

$$A_{abc} \frac{\partial V}{\partial \phi_a} + B_{ab} \frac{\partial^2 V}{\partial \phi_a \partial \phi_c} + B_{ac} \frac{\partial^2 V}{\partial \phi_a \partial \phi_b} = 0.$$



Further examination shows that in the case of  $a_n^{(1)}$  that the charge

$$P_2 = \int_{-\infty}^0 dx^1 (T_{+3} - \Theta_{+1} + T_{-3} - \Theta_{-1}) - \Sigma_2$$

is conserved on the half line if the boundary potential is chosen as

$$\mathcal{B} = \frac{m}{\beta^2} \sum_{i=0}^n A_i e^{\frac{\beta}{2} \alpha_i \cdot \phi}$$

with either all  $A_i = 0$  or  $A_i^2 = 4$ . The boundary contribution in this case

$$\Sigma_2 = -\sqrt{2} B_{ab} \partial_0 \phi_a \mathcal{B}_b.$$

The next higher charge does not reveal any new restrictions but exactly the same [86]. For the  $d_n^{(1)}$  theory one finds conditions leading to (2.15) by investigating spin  $\pm 3$  charges and for  $e_6^{(1)}$  one has to look at spin  $\pm 4$  charges. For a full list of boundary conditions see [89].

However, this approach does not prove that the theory is integrable at all as one needs the conservation of infinitely many quantities which are involution. This problem was solved in [87]. A Lax-pair representation is given and it is shown that (2.15) is the most general boundary condition guaranteeing the integrability of affine Toda field theory with real coupling for the *ade* series with the exception of the sinh-Gordon theory.

### 2.3.2 Solutions for the (classical) reflection Bootstrap Equation

In this section reflection factors satisfying the classical bootstrap equation (2.12) will be presented. Also, they will be shown to be classical limits of solutions to the reflection bootstrap equation (2.7). The calculations can be found in [86,87].

#### 2.3.2.1 Symmetric Boundary Conditions

With the restrictions (2.15) the equations of motion for affine Toda field theory on the half line are

$$\begin{aligned} \partial^2 \phi &= -\frac{m^2}{\beta} \sum_{i=0}^n n_i \alpha_i e^{\beta \alpha_i \cdot \phi} \quad x^1 < 0, \\ \partial_1 \phi &= -\frac{m}{2\beta} \sum_{i=0}^n A_i \alpha_i e^{\frac{\beta}{2} \alpha_i \cdot \phi} \quad x^1 = 0. \end{aligned} \tag{2.16}$$

The total energy of a solution to the half-line affine Toda field theory is given by

$$E = \int_{-\infty}^0 \mathcal{E} dx + B$$

where

$$\mathcal{E} = \frac{1}{2}(\dot{\phi}^2 + \phi'^2) + \frac{m^2}{\beta^2} \sum_{i=0}^r (e^{\beta \alpha_i \cdot \phi} - 1).$$

If zero energy solutions exist, bound states are allowed for  $B < 0$  as the total energy would be negative and could be trapped as a small oscillation. The solution  $\phi$  can be written as an expansion in the coupling constant  $\beta$

$$\phi = \sum_{i=-1}^{\infty} \beta^i \phi^{(i)}. \quad (2.17)$$

It is necessary to start the expansion at  $i = -1$  because the right hand side of the boundary condition in (2.16) might not be zero for the  $\beta^{-1}$  contribution. Now it is possible to expand the equations of motions (2.16) and for the first two terms they are

$$\begin{aligned} \partial^2 \phi^{(-1)} &= -m^2 \sum_{i=0}^n n_i \alpha_i e^{\alpha_i \cdot \phi^{(-1)}} \quad x^1 < 0, \\ \partial_1 \phi^{(-1)} &= -\frac{m}{2} \sum_{i=0}^n A_i \alpha_i e^{\frac{1}{2} \alpha_i \cdot \phi^{(-1)}} \quad x^1 = 0, \end{aligned} \quad (2.18)$$

$$\begin{aligned} \partial^2 \phi^{(0)} &= -m^2 \sum_{i=0}^n n_i \alpha_i e^{\alpha_i \cdot \phi^{(-1)}} \alpha_i \cdot \phi^{(0)} \quad x^1 < 0, \\ \partial_1 \phi^{(0)} &= -\frac{m}{4} \sum_{i=0}^n A_i \alpha_i e^{\frac{1}{2} \alpha_i \cdot \phi^{(-1)}} \alpha_i \cdot \phi^{(0)} \quad x^1 = 0. \end{aligned} \quad (2.19)$$

As remarked before the  $a_n^{(1)}$  case (2.18) has the solution  $\phi^{(-1)} = 0$  for symmetric boundary conditions, i.e.  $A_i = A$  for all  $i = 0, \dots, n$  where  $A \in \{2, -2, 0\}$ . This represents a static “ground-state” solution of lowest energy and a quantum theory could be constructed in terms of perturbations around this basic solution. If  $\phi^{(-1)} = 0$  is not a possibility one might still be able to find a static solution of least energy which would serve as an effective potential for the solution of (2.19). This will be the case for asymmetric boundary conditions of the  $a_2^{(1)}$  theory.

The solution of (2.19) is, in terms of the eigenvectors  $\rho_a$  of the mass matrix  $M^2$

$$\phi^{(0)} = \sum_{a=1}^n \rho_a (R_a e^{-ip_a x^1} + I_a e^{ip_a x^1}) e^{-i\omega_a x^0}$$

where

$$M^2 \rho_a = m^2 \sum \alpha_i \otimes \alpha_i \rho_a = m_a^2 \rho_a, \quad \omega_a^2 - p_a^2 = m_a^2.$$

The reflection factor is given by the ratio

$$K_a = \frac{R_a}{I_a} = \frac{ip_a + Am_a^2/4m}{ip_a - Am_a^2/4m}. \quad (2.20)$$

For the boundary condition  $A = 0$  the reflection factor is 1. For the cases  $A^2 = 4$  the reflection factor has poles at

$$p_a = -i \frac{Am^2}{4m}.$$

The masses of the  $a_n^{(1)}$  affine Toda field theory are  $m_a = 2m \sin(\frac{a\pi}{n+1})$ . For the boundary condition  $A_i = -2$  the potential (2.14) is negative and allows bound states with the masses given by

$$\omega_a^2 = m^2 \sin^2 \left( \frac{2a\pi}{n+1} \right)$$

for the  $a$ -th channel. The corresponding solution to the linear problem decays exponentially away from the boundary as  $x^1 \rightarrow -\infty$ . For  $n$  even the masses are doubly degenerate, whereas for  $n$  odd there is a four-fold degeneracy and  $\omega_{(n+1)/2} = 0$ .

### 2.3.2.2 A connection between solutions of the classical and quantum reflection bootstrap equation

To keep things simple consider the  $a_2^{(1)}$  theory whose two conjugate particles have the masses

$$m_1 = m_2 = \sqrt{3}m.$$

In the block notation introduced earlier (1.29) the reflection factor for the negative boundary condition can be written as

$$K_a = \frac{ip - \frac{3m}{2}}{ip + \frac{3m}{2}} = -(1)(2), \quad \text{with } p = \sqrt{3}m \sinh \theta. \quad (2.21)$$

One of the  $S$ -matrices of this theory is  $S_{11}(\theta) = \frac{(2)}{(B)(2-B)}$ . Now, one should observe that the boundary conditions do not distinguish between the two particles. So the reflection matrices should be the same for both particles. Also, it has just been shown

that the classical reflection factor has a simple pole at  $\theta = i\pi/3$ . In the full line theory the quantum particle spectrum of the simply laced theories is essentially the same as the classical one. Assuming the same happens for the half-line a “minimal” solution for a reflection matrix solving (2.7) and (2.6) is

$$K_1^0(\theta) = K_2^0(\theta) = -\frac{(1)(2 + \frac{B}{2})}{(\frac{B}{2})}. \quad (2.22)$$

The classical limit for this matrix is the reflection factor (2.21). So in contrast to the solution of Fring and Köberle [80,81] and Sasaki [79] this reflection matrix is not the unit matrix in the classical limit. Also, note that the expression (2.22) is not invariant under the weak strong coupling transformation  $\beta \rightarrow \frac{4\pi}{\beta}$  like the  $S$ -matrix.

A further discussion of this case and generalisation of (2.22) for  $a_n^{(1)}$  can be found in [85,86]. Also, it is worth mentioning that the reflection factor for  $A_i = 2$  is

$$\tilde{K}_a(\theta) = \frac{1}{K_a(\theta)} = -(-1)(-2).$$

The reflection matrix is expected to have no bound states since there are no classical bound states because  $B > 0$  and a quick check shows that a solution fulfilling the reflection bootstrap (2.7) and the classical limit is

$$\tilde{K}_1^0(\theta) = \tilde{K}_2^0(\theta) = -\frac{(3 - \frac{B}{2})}{(2)(1 - \frac{B}{2})}.$$

### 2.3.2.3 Asymmetric Boundary Conditions for $a_2^{(1)}$

For all affine Toda field theories, except  $a_n^{(1)}$  with symmetrical boundary conditions,  $\phi^{(-1)} = 0$  is only a solution for natural Neumann boundary conditions. So, looking at  $a_2^{(1)}$  with asymmetrical boundary conditions might be a place learn some techniques which are useful for other theories, especially the  $a_2^{(2)}$  theory later on (The following calculation can be found in [86].).

The solution to (2.18) will provide a non-trivial static background potential for the solution of the linear scattering problem (2.19). These solutions are related to static single soliton solutions in the imaginary coupling theory (1.5). Their singularities will be chosen to lie in the positive real  $x$ -axis.

A choice for asymmetrical boundary conditions is  $A_1 = 2$ ,  $A_2 = A_0 = -2$ . With the ansatz

$$\phi^{(-1)}(x, t) = \alpha_1 \rho(x) = \alpha_1 \rho$$

which is compatible with the boundary condition equation (2.18) turns into the time independent Bullough-Dodd equation (1.14)

$$\begin{aligned} \rho'' &= e^{2\rho} - e^{-\rho} \quad x < 0 \\ \rho' &= -(e^\rho - e^{-\frac{\rho}{2}}) \quad x = 0. \end{aligned} \tag{2.23}$$

Integrating this equation once yields the following differential equation

$$(\rho')^2 = e^{2\rho} + 2e^{-\rho} + 3,$$

which can be used at the boundary  $x = 0$  to give with the second equation in (2.23)

$$(e^\rho - e^{-\frac{\rho}{2}})^2 = e^{2\rho} + 2e^{-\rho} + 3.$$

This equation has three solutions for  $e^{\rho/2}$

$$e^{\rho/2} = \begin{cases} -1 \\ \frac{1}{2} \\ \rho \rightarrow \infty \end{cases}, \text{ i.e. a singularity at } x = 0.$$

Only the second solution is without any problems. The relevant solution\* of the Bullough-Dodd equation ( $\rho \rightarrow 0$  for  $x \rightarrow -\infty$ ) is (1.40)

$$e^{-\rho} = \frac{1 + 4E + E^2}{(1 - E)^2} = 1 + \frac{3/2}{\sinh^2 \sqrt{3}(x - x_0)/2}, \quad E = e^{\sqrt{3}(x - x_0)}. \tag{2.24}$$

The parameter  $x_0$  is determined by the boundary condition and must satisfy  $x_0 > 0$  because the singularity of (2.24) is otherwise not in the positive half of the  $x$ -axis.

The positive solution of

$$\coth^2 \sqrt{3}x_0/2 = 3$$

satisfies this condition. Now, one should turn to the equations for  $\phi^{(0)}$  (2.19). They are with the choice made for  $\phi^{(-1)}$

$$\begin{aligned} \partial^2 \phi^{(0)} &= - \begin{pmatrix} 2e^{2\rho} - e^{-\rho} & 0 \\ 0 & 3e^{-\rho} \end{pmatrix} \phi^{(0)} \quad x^1 < 0 \\ \partial_1 \phi^{(0)} &= -\frac{1}{2} \begin{pmatrix} 2e^{2\rho} - e^{-\rho/2} & 0 \\ 0 & -3e^{-\rho/2} \end{pmatrix} \phi^{(0)} = \begin{pmatrix} 3/4 & 0 \\ 0 & 3 \end{pmatrix} \phi^{(0)} \quad x^1 = 0. \end{aligned} \tag{2.25}$$

---

\* This solution is similar to the soliton solutions first mentioned by Aratyn et al. [90].

First the second component of  $\phi^{(0)}$  will be solved. As in the case with symmetric boundary conditions the expected form of  $\phi^{(0)}$  is

$$\phi^{(0)} = e^{-i\omega t} \Phi(x).$$

Changing to the variable  $z = \sqrt{3}x/2$ ,  $\Phi(x)$  has to satisfy

$$\Phi''(z) = \left( -\lambda^2 + \frac{6}{\sinh^2(z - z_0)} \right) \Phi(z).$$

Also, it is convenient to set

$$\lambda^2 = (4/3)(\omega^2 - 3) = (4/3)p^2 = 4 \sinh^2 \theta \quad (2.26)$$

A solution to this equation can be written as [91]

$$\Phi_L(z) = \left( \frac{d}{dz} - 2 \coth(z - z_0) \right) \left( \frac{d}{dz} - \coth(z - z_0) \right) e^{i\lambda z} \quad (2.27)$$

and the general solution is then

$$\Phi(z) = a\Phi_L(z) + a^*\Phi_L^*(z).$$

The reflection factor can be read off the following expression for  $\Phi$  as  $z \rightarrow -\infty$

$$\Phi \approx a(i\lambda + 2)(i\lambda + 1)e^{i\lambda z} + a^*(-i\lambda + 2)(-i\lambda + 1)e^{-i\lambda z} = Ie^{i\lambda z} + Re^{-i\lambda z}.$$

Which yields (2.20)

$$K = \frac{R}{I} = \frac{a^*(-i\lambda + 2)(-i\lambda + 1)}{a(i\lambda + 2)(i\lambda + 1)}.$$

The ratio  $\frac{a^*}{a}$  is determined by the boundary condition. One needs to work out the values of  $\Phi_L(0)$  and  $\Phi_L'(0)$

$$\begin{aligned} \Phi_L(0) &= (i\lambda)^2 + 3\sqrt{3}(i\lambda) + 8 \\ \Phi_L'(0) &= (i\lambda)^3 + 3\sqrt{3}(i\lambda)^2 + 14(i\lambda) + 12\sqrt{3}. \end{aligned}$$

This has to fulfil (2.25) and therefore

$$a((i\lambda)^3 + \sqrt{3}(i\lambda)^2 - 4(i\lambda) - 4\sqrt{3}) + c.c. = a(i\lambda + \sqrt{3})((i\lambda)^2 - 4) + c.c. = 0$$

implying that the ratio is

$$\frac{a^*}{a} = \frac{i\lambda + \sqrt{3}}{i\lambda - \sqrt{3}}.$$

Remembering (2.26) the reflection coefficient can be written with  $\lambda = 2s$

$$K = \frac{is + \sqrt{3}/2}{is - \sqrt{3}/2} \frac{is - 1}{is + 1} \frac{is - 1/2}{is + 1/2}.$$

In the usual block notation (1.29) this is

$$K_1 = \frac{(1/2)(3/2)^2(5/2)}{(1)(2)(3)}. \quad (2.28)$$

Since  $(3) = -1$  the denominator is the same as in the symmetric boundary case. Also,  $K_1$  satisfies the classical bootstrap equation (2.12) which implies that the other channel has to have the same reflection factor. Using the observation that the denominator is the same in the symmetric case it easy to write an extrapolation of a coupling dependent reflection matrix which satisfies the bootstrap equation and has (2.28) as its classical limit

$$K_1 = K_2 = (1/2)(3/2)^2(5/2) \frac{(3 - \frac{B}{2})}{(1 - \frac{B}{2})(2)}.$$

But this is not the only possibility. For any function  $C(\beta)$  which vanishes for  $\beta = 0$  there is another reflection matrix

$$K_1 = K_2 = (1/2 + C)(3/2 - C)(3/2 + C)(5/2 - C) \frac{(3 - \frac{B}{2})}{(1 - \frac{B}{2})(2)}$$

which solves all necessary equations.

One way to check the above calculation is to work out the reflection for the other channel directly from the equations of motions. As already said, the result should be the same due to the classical reflection bootstrap equation.

The only difference is the linear approximation in the linear background potential which has the following form for the second channel

$$\Phi''(z) = \left( -\lambda^2 + \frac{4r}{q^2} \right) \Phi,$$

where

$$\begin{aligned} r &= -6E(1 - 6E + 3E^2 + 4E^3 + 3E^4 - 6E^5 + E^6) \\ q &= (1 + 4E + E^2)(1 - E)^2, \quad E = e^{2(z-z_0)}. \end{aligned}$$

This time the solution  $\Phi$  takes the form

$$\Phi_L(z) = \frac{p}{q} e^{i\lambda z}$$

where  $\lambda$  satisfies (2.26) and the general solution of  $\Phi$  is given by (2.27), the function  $p$  depends on  $\lambda$  and is up to an overall factor

$$p = (2 + i\lambda)(1 + i\lambda) - 2(\lambda^2 + 4)(E + E^3) + 6(2 + \lambda^2)E^2 + (2 - i\lambda)(1 - i\lambda)E^4.$$

For  $z = 0$  the following boundary condition is given by (2.25)

$$\Phi'(0) = \frac{\sqrt{3}}{2} \Phi(0).$$

One can calculate the ratio  $\frac{a^*}{a}$  again and finds

$$\frac{a^*}{a} = \frac{i\lambda + \sqrt{3}}{i\lambda - \sqrt{3}}.$$

Altogether one gets the same result for the reflection factor as in (2.28). It is surprising that the classical reflection factors obtained here obey the classical reflection bootstrap (2.12). This is true for other theories, as for example  $d_5^{(1)}$ , as shown in [86].

#### 2.3.2.4 The sinh-Gordon Model

Since this model has boundary conditions with a continuous parameter, as will be seen has the Bullough-Dodd model, it will be instructive to repeat the analysis in the context of the method just applied to other Toda models. This discussion can also be found in [86]. In the disguise of the sine-Gordon model there is already a lot of information in the literature.



The static background solution will again be called  $\rho = \phi^{(-1)}$  and it has to satisfy the equations

$$\begin{aligned}\rho'' &= -\sqrt{2}(e^{\sqrt{2}\rho} - e^{-\sqrt{2}\rho}) \quad x < 0 \\ \rho' &= -\sqrt{2}(\epsilon_1 e^{\rho/\sqrt{2}} - \epsilon_0 e^{-\rho/\sqrt{2}}) \quad x = 0.\end{aligned}\tag{2.29}$$

Integrating the first equation the boundary equation implies

$$\begin{aligned}\rho' &= \sqrt{2}(\epsilon_1 e^{\rho/\sqrt{2}} - \epsilon_2 e^{-\rho/\sqrt{2}}) \quad x < 0 \\ e^{\sqrt{2}\rho} &= \frac{1 + \epsilon_0}{1 + \epsilon_1} \quad x = 0.\end{aligned}$$

So, assuming  $\epsilon_0 > \epsilon_1$  (otherwise shift  $x_0$  in the solution by  $i\pi/2$ ) with  $\coth x_0 = \sqrt{\frac{1+\epsilon_0}{1+\epsilon_1}}$  the ground state is

$$e^{\sqrt{2}\rho} = \frac{1 + e^{2(x-x_0)}}{1 - e^{2(x-x_0)}}.$$

Then the linearised wave equation in this background is

$$\begin{aligned}\partial^2 \phi^{(0)} &= -4\left(1 + \frac{2}{\sinh^2 2(x-x_0)}\right)\phi^{(0)} \quad x < 0 \\ \partial_1 \phi^{(0)} &= -(\epsilon_0 \tanh x_0 + \epsilon_1 \coth x_0)\phi^{(0)} \quad x = 0.\end{aligned}\tag{2.30}$$

It is possible to compute the reflection coefficient in terms of the parameters of the boundary potential. Again write  $\phi^{(0)} = e^{-i\omega t}\Phi(z)$  which allows one to express the solution as

$$\Phi(z) = a(i\lambda - \coth(z - z_0))e^{i\lambda z} + c.c., \quad \lambda = \sinh \theta,$$

the ratio  $a^*/a$  is again determined by the boundary condition. Using the parametrisation

$$\epsilon_i = \cos a_i \pi, \quad |a_i| \leq 1, \quad i = 0, 1\tag{2.31}$$

the reflection factor can be written as

$$K = -(1)^2[(1 + a_0 + a_1)(1 - a_0 + a_1)(1 + a_0 - a_1)(1 - a_0 - a_1)]^{-1}\tag{2.32}$$

An extension beyond the limits of  $a_i$  in (2.31) is achieved by  $a_i \rightarrow a_i + 2$ . The reflection factor is similar to the reflection matrix given by Ghoshal [72] of the lightest sine-Gordon breather if one takes a suitable limit for the classical case after analytic continuation of  $\beta$ .

In this case one can give an example of how to check the stability of the background potential by examining the energy of the solution  $\Phi^{(-1)} = \rho$  [92]. The energy is given by

$$E = \int_{-\infty}^0 dx \left( \frac{(\phi')^2}{2} + (e^{\sqrt{2}\rho} - e^{-\sqrt{2}\rho} + 2) \right) + A_1 e^{\rho_0/\sqrt{2}} + A_0 e^{-\rho_0/\sqrt{2}}.$$

Using the Bogomolny argument this can be rewritten by replacing the integrand with

$$\frac{1}{2} \left( \rho' - \sqrt{2}(e^{\sqrt{2}\rho} - e^{-\sqrt{2}\rho}) \right)^2 + \sqrt{2}\rho'(e^{\sqrt{2}\rho} - e^{-\sqrt{2}\rho}),$$

yielding

$$E \geq -4 + (A_0 + 2)e^{-\rho_0/\sqrt{2}} + (A_1 + 2)e^{\rho_0/\sqrt{2}}.$$

So, provided  $A_0$  and  $A_1$  are at least  $-2$ , the energy is bounded below which indicates a stability of the solution.

## 2.4 Reflection factors of $a_2^{(2)}$ theory

In this section reflection factors for the  $a_2^{(2)}$  theory will be calculated [93] along the lines of the previous section [86]. This theory is the simplest example for the self-dual non-simply laced theories. Though the calculation will be similar to the one for  $a_2^{(1)}$  there will be a difference in the boundary condition which allows in this case a continuous parameter. Therefore it is continuously connected to the Neumann boundary condition. Both Fring and Köberle [80] and Kim [83] give reflection matrices for this model and they will be compared with the results presented here.

The bulk theory has been met before (1.14) in the first chapter and for  $m^2 = 2$  and  $\beta = \sqrt{2}$  the Lagrangian is

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - (e^{2\phi(x,t)} + 2e^{-\phi(x,t)} - 3). \quad (2.33)$$

The equation of motion is given by

$$\partial^2 \phi(x, t) = -2 \exp(2\phi(x, t)) + 2 \exp(-\phi(x, t)). \quad (2.34)$$

In [87] the boundary condition is given as

$$\mathcal{B} = A_1 \exp(\phi(x, t)) + A_0 \exp\left(-\frac{\phi(x, t)}{2}\right). \quad (2.35)$$

With (2.35) the boundary condition is therefore

$$\partial_x \phi(x, t)|_{x=0} = -A_1 \exp(\phi(0, t)) + \frac{A_0}{2} \exp\left(-\frac{\phi(0, t)}{2}\right), \quad (2.36)$$

where  $A_0(A_1^2 - 2) = 0$ .

In [87] this condition is derived from the spin  $\pm 6$  charge, insisting that the combination  $T_6 - T_{-6} + \Theta_4 - \Theta_{-4}$  is a total time derivative in the presence of the boundary term. Explicit expressions for the densities are

$$T_{\pm 6} = (\partial_{\pm} \phi)^6 - 5(\partial_{\pm} \phi)^3 \partial_{\pm}^3 \phi + 5(\partial_{\pm}^2 \phi)^3 + 3(\partial_{\pm}^3 \phi)^2$$

and

$$\Theta_{\pm 4} = -\frac{1}{8} \left[ 4(\partial_{\pm} \phi)^2 \partial_{\pm}^2 \phi (-15V' + 6V''') + 12(\partial_{\pm}^2 \phi)^2 V'' + (\partial_{\pm} \phi)^4 (10V'' - 6V''') \right]$$

such that they satisfy

$$\partial_{\pm} T_{\pm 6} = \partial_{\pm} \Theta_{\pm 4}$$

with

$$V(\phi) = e^{2\phi} + 2e^{-\phi}.$$

The approach via the Lax-pair gives the same result. There is, however, a different approach to the problem of finding integrable boundary conditions [57]. This approach tries to identify conditions under which symmetries of the equation are still conserved. However this approach hasn't been applied to affine Toda field theory in general. But, in the case of the Bullough-Dodd equation, Habibullin finds the same condition for the boundary.

#### 2.4.1 Solutions to the classical Reflection Bootstrap Equation

As in the cases discussed before, one way to find potential solutions to the classical reflection bootstrap is to expand the solution  $\phi$  in powers of the coupling constant (2.17) and work out reflection factors of the linear approximation in the static background. For the  $a_2^{(2)}$  the equations of motion (2.18) and (2.19) for the first two terms are  $\phi^{(-1)} = \Phi_0$  and  $\phi^{(0)} = \Phi_1$

$$\partial^2 \Phi_0(x, t) = -2 \exp(2\Phi_0(x, t)) + 2 \exp(-\Phi_0(x, t)) \quad (2.37a)$$

$$\partial_x \Phi_0(x, t)|_{x=0} = -A_1 \exp(\Phi_0(0, t)) + \frac{A_0}{2} \exp\left(-\frac{\Phi_0(0, t)}{2}\right), \quad (2.37b)$$

and

$$\partial^2 \Phi_1(x, t) = - \left[ 4 \exp(2\Phi_0(x, t)) + 2 \exp(-\Phi_0(x, t)) \right] \Phi_1(x, t) \quad (2.38a)$$

$$\partial_x \Phi_1(x, t)|_{x=0} = - \left[ A_1 \exp(\Phi_0(0, t)) + \frac{A_0}{4} \exp\left(-\frac{\Phi_0(0, t)}{2}\right) \right] \Phi_1(x, t), \quad (2.38b)$$

where, in both cases,  $A_0(A_1^2 - 2) = 0$ . Due to these conditions the discussion will have to be split into several cases later on. First some common features of the background potential and the linear approximation will be shown.

#### 2.4.1.1 General features of the background potential

Unless one chooses the natural Neumann boundary conditions  $\Phi_0 = 0$  is not a valid solution for (2.37) because of the boundary condition. As in the  $a_2^{(1)}$  case the background solution  $\Phi_0$  should be constant in time and of least energy. Therefore assume  $\Phi_0(x, t) = \Phi_0(x)$  and (2.37a) reduces to

$$\partial_x^2 \Phi_0(x) = 2 \exp(2\Phi_0(x)) - 2 \exp(-\Phi_0(x)) \text{ for } \leq 0. \quad (2.39)$$

This is again the time independent Bullough-Dodd equation (2.23). Its solutions have been calculated in the introduction of the thesis and are for the static case

$$e^{-\Phi_0(x)} = \begin{cases} 1 + \frac{\frac{3}{2}}{\sinh^2 \frac{\sqrt{6}}{2}(x-x_0)} = \frac{1+4E+E^2}{(1-E)^2} & \text{sinh solution} \\ 1 - \frac{\frac{3}{2}}{\cosh^2 \frac{\sqrt{6}}{2}(x-x_0)} = \frac{1-4E+E^2}{(1+E)^2} & \text{cosh solution} \end{cases} \quad (2.40)$$

with  $E = \exp(\sqrt{6}(x - x_0))$ . The solutions will be called sinh or cosh solution respectively in this chapter. Note that the transformation  $E \rightarrow -E$  connects both solutions. Unlike the cases discussed before the cosh solution will actually be used for this model. For later reference it is useful to list the properties of the two solutions.

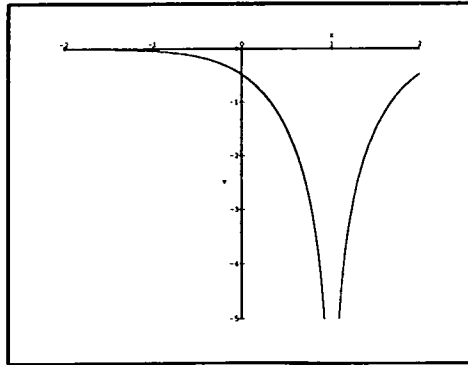


Fig. 2.4: The sinh solution  $\Phi_0(x)$

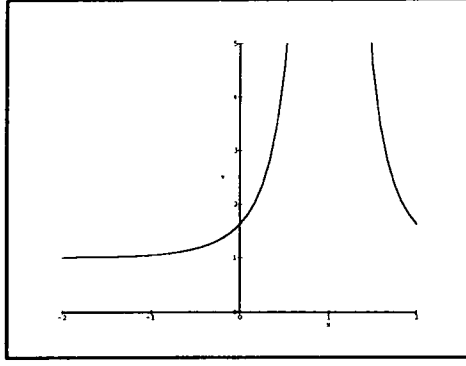


Fig. 2.5: The sinh solution plotted as  $e^{-\Phi_0(x)}$

- The sinh-solution has  $\Phi_0(x) < 0$ . Also  $x_0$  has to be positive in order to have the singularity outside the negative half-line. From the graph one easily reads off that  $0 \leq e^{\Phi_0(x)} < 1$  and  $e^{-\Phi_0(x)} > 1$ , for the sinh solution.
- The cosh solution  $e^{-\Phi_0(x)}$  has no real singularities except at  $x_{\pm}$ , given by  $x_{\pm} = x_0 \pm \sqrt{\frac{2}{3}} \text{arcosh} \sqrt{\frac{3}{2}}$ ,  $e^{-\Phi_0(x_{\pm})} = 0$ . Therefore the solution  $\Phi_0(x)$  has singularities at those points. In order to avoid a singularity in the negative half-line  $x_-$  has to be chosen appropriately. This will be discussed in detail later. In contrast to the sinh solution, for the cosh solution  $e^{\Phi_0(x)} > 1$  and  $0 < e^{-\Phi_0(x)} < 1$ . Also for the cosh solution,  $\Phi_0(x)$  is always positive, or complex between the singularities.

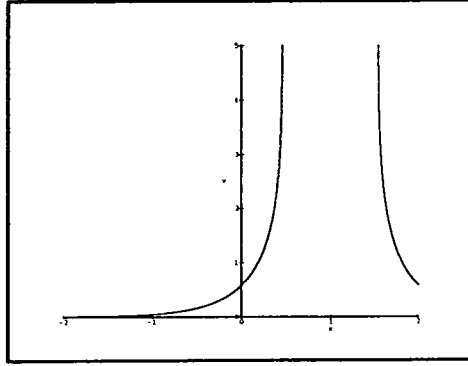


Fig. 2.6: The cosh solution  $\Phi_0(x)$

The values of  $x_0$  are determined by the boundary condition and will be calculated in the detailed discussion of both cases later on.

Again it is possible to integrate (2.39) once with the assumption that  $\Phi_0(x)$  vanishes for  $x \rightarrow -\infty$  the result is

$$(\partial_x \Phi(x))^2 = 2 \exp(2\Phi_0(x)) + 4 \exp(-\Phi_0(x)) - 6 \quad \text{for } x \leq 0. \quad (2.41)$$

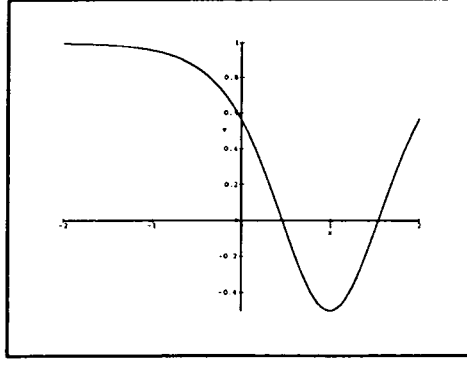


Fig. 2.7: The cosh solution, plotted as  $e^{-\Phi_0(x)}$

Squaring equation (2.37b) and using the above result gives the following relation for  $x = 0$ , where  $e^{\Phi_0} = e^{\Phi_0(0)}$ ,

$$(A_1^2 - 2)e^{2\Phi_0} - A_0 A_1 e^{\Phi_0/2} + \frac{A_0^2 - 16}{4} e^{-\Phi_0} + 6 = 0. \quad (2.42)$$

This equation will simplify later on.

#### 2.4.1.2 General features of the solution in the background potential

The solution to the linear background equation (2.38a) is the same for all boundary conditions. The actual computation will be very similar to the one for  $a_2^{(1)}$  with asymmetric boundary conditions. The reflection coefficient will be determined from its solution and the specific boundary condition. In the following calculation the sinh solution will be used when  $e^{\Phi_0}$  has to be expressed in terms of  $E$ .

Assume that the solution in the background potential factorises into an  $x$  dependent and a  $t$  dependent term  $\Phi_1(x, t) = \Psi(x)e^{i\omega t}$ . Then (2.38a) simplifies to

$$\Psi(x)'' = [-\omega^2 + 4e^{2\Phi_0} + 2e^{-\Phi_0}] \Psi(x). \quad (2.43)$$

Again one introduces the variable  $\lambda$  depending on the rapidity

$$\omega^2 - 6 = 6 \sinh^2 \theta = 6\lambda^2 \quad (2.44)$$

and rewrites the equation as

$$\Psi(x)'' = [-6\lambda^2 - 6 + 4e^{2\Phi_0} + 2e^{-\Phi_0}] \Psi(x).$$

Now it is useful to change variables  $z = x\sqrt{6}$  such that  $E$  changes to  $E = e^{z-z_0}$ . The potential for this linear scattering problem is given by

$$V(z) = -1 + \frac{1}{3}(2e^{2\Phi_0(z)} + e^{-\Phi_0(z)}) = \frac{p}{q^2}$$

where

$$\begin{aligned} r &= -6E(1 - 4E - 6E^2 - 4^3 + E^4) \\ q &= 1 + 3E - 3E^2 - E^3. \end{aligned}$$

The equations (2.38a, b) are, in these new variables,

$$\Psi(z)'' = [-\lambda^2 + \frac{r}{q^2}]\Psi(z) \quad (2.45a)$$

$$\Psi(z)'|_{z=0} = -\frac{1}{4\sqrt{6}}[4A_1e^{\Phi_0} + A_0e^{-\Phi_0/2}]\Psi(z). \quad (2.45b)$$

The differential equation (2.45a) is solved by

$$\Psi(z) = a\frac{p}{q}e^{i\lambda z} + a^*\frac{p^*}{q^*}e^{-i\lambda z}$$

where  $a$  is a complex number determined by the boundary condition and

$$\begin{aligned} p &= (1 + i2\lambda)(1 + i\lambda) - 3E(1 - i2\lambda)(1 + i\lambda) \\ &\quad + 3E^2(1 + i2\lambda)(1 - i\lambda) - E^3(1 - i2\lambda)(1 - i\lambda). \end{aligned}$$

By looking at  $\Psi(z)$  as  $z \rightarrow -\infty$  the reflection coefficient is found to be (2.20)

$$K = \frac{a^*}{a} \frac{(1 - i2\lambda)(1 - i\lambda)}{(1 + i2\lambda)(1 + i\lambda)}$$

as  $p \rightarrow (1 + i2\lambda)(1 + i\lambda)$  and  $q \rightarrow 1$ . Using the block notation (1.29) this can be written as

$$K = \frac{a^*}{a}(1)(3)^2(5) \quad (2.46)$$

where one should keep in mind that even though the Coxeter number for  $a_2^{(2)}$  is  $h = 3$  it is better to use  $h^{(2)} = 2 \cdot 3 = 6$  because it is a non-simply laced algebra, to get

integer values for the poles of the  $S$ -matrix [25,94]\*. Using the explicit formula of the solution  $\Psi(z)$  in terms of  $p$  and  $q$  equation (2.45b) can now be written as

$$\begin{aligned}\Psi'(z) &= \left( i\lambda a \frac{p}{q} + \frac{ap'}{q} - \frac{apq'}{q^2} \right) \Big|_{z=0} + c.c. \\ &= -\frac{1}{4\sqrt{6}} \left( 4A_1 e^{\Phi_0} + A_0 e^{-\Phi_0/2} \right) \left( a \frac{p}{q} \Big|_{z=0} + c.c. \right).\end{aligned}\tag{2.47}$$

This equation determines the ratio  $\frac{a^*}{a}$

$$\frac{a^*}{a} = -\frac{i\lambda pq + p'q - pq' + \frac{1}{4\sqrt{6}}(4A_1 e^{\Phi_0} + A_0 e^{-\Phi_0/2})pq}{-i\lambda p^*q + p'^*q - p^*q' + \frac{1}{4\sqrt{6}}(4A_1 e^{\Phi_0} + A_0 e^{-\Phi_0/2})p^*q}.\tag{2.48}$$

The expression for the cosh solution can be gained by replacing  $E$  by  $-E$  in  $p, p', q, q'$  and  $e^{\Phi_0}$ .

#### 2.4.1.3 Two Cases

Now it is time to look at the condition which the coefficients  $A_0$  and  $A_1$  of the boundary potential (2.35) have to fulfil

$$A_0(A_1^2 - 1) = 0.$$

The solutions to this equation can be split into two cases

- $A_0 = 0, A_1 \in \mathbb{R}$
- $A_1^2 = 2, A_0 \in \mathbb{R}.$

The first case contains the natural Neumann condition. Kim [83] has calculated reflection factors fulfilling the bootstrap for this boundary condition. However, his solution tends to unity in the classical limit and it will not be possible to say whether it corresponds to any of the solutions presented here.

*Case A:  $A_0 = 0$*

For  $A_0 = 0$  equation (2.42) simplifies to

$$(A_1^2 - 2)e^{\Phi_0} - 4e^{-\Phi_0} + 6 = 0.\tag{2.49}$$

---

\* A comprehensive list of Coxeter numbers for all affine Toda field theories can be found in [34].



With  $g = e^{\Phi_0}$  this can be written as a cubic equation in  $p$

$$f(g) = (A_1^2 - 2)g^3 + 6g - 4 = 0. \quad (2.50)$$

It was mentioned before in (2.40) that the sinh solution requires  $0 < e^{\Phi_0} < 1$  for all  $x \leq 0$  and the cosh solution  $e^{\Phi_0} > 1$  for all  $x \leq 0$ . Because  $g = e^{\Phi_0(0)}$  a discussion of the polynomial  $f(g)$  is going to help to decide which solution matches a particular boundary condition. The polynomial has turning points for  $g^2 = \frac{2}{2-A_1^2}$ . There are two different situations

- If  $A_1^2 > 2$  there are no real turning points. This means there is a unique solution to (2.50) with  $g > 0$  because  $f(0)$  is negative and the factor of  $g^3$  positive. Also  $f(1) = A_1^2 > 0$  therefore the root  $p$  of (2.50) has to satisfy  $0 < g = \exp(\Phi_0) < 1$  which indicates that the sinh solution is relevant

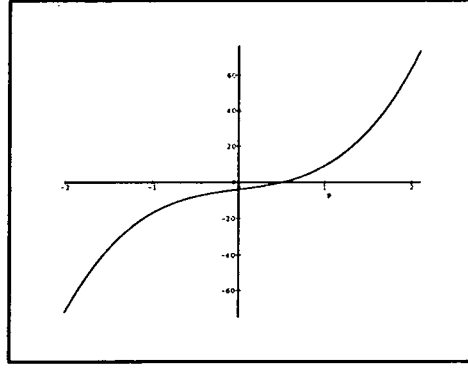


Fig. 2.8: A plot of  $f(g)$  for  $A_1 = 3$ (eq.(2.50))

((2.8) illustrates this with a plot of  $f(g)$  for  $A_1 = 3$ .)

- If  $0 < A_1^2 < 2$  there are two real turning points at  $p_{\pm} = \pm \sqrt{\frac{2}{2-A_1^2}}$ . The values of the polynomial at these points are  $f(g_{\pm}) = 4(\pm \sqrt{\frac{2}{2-A_1^2}} - 1)$ . So, the value at the positive turning point  $g_+$  is always positive,  $g_+ > 0$ . Again  $f(0) = -4$  and because  $f(p)$  tends to  $-\infty$  for  $g \rightarrow \infty$  there must be two positive roots  $g_1, g_2$ . Furthermore  $f(1) = A_1^2 > 0$  so one root is smaller than one,  $p_1 < 1$  corresponding to the sinh solution, and the other one is bigger than one,  $g_2 > 1$  corresponding to the cosh solution

((2.9) illustrates this with a plot of  $f(g)$  for  $A_1 = 1$ .)

*The reflection coefficient*

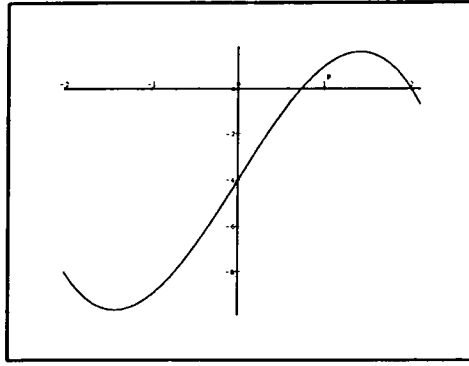


Fig. 2.9: Figure 2: A plot of  $f(g)$  for  $A_1 = 1$ (eq.(2.50))

To determine the reflection factor equation (2.48) has to be solved. For  $A_0 = 0$  it simplifies to

$$\frac{a^*}{a} = - \frac{i\lambda pq + p'q - pq' + \frac{1}{\sqrt{6}}A_1 e^{\Phi_0} pq}{-i\lambda p^*q + p'^*q - p^*q' + \frac{1}{\sqrt{6}}A_1 e^{\Phi_0} p^*q}. \quad (2.51)$$

In the equations above  $p, q, p', q', \Phi_0$  have been evaluated at  $z = 0$ .

If one parametrises the coefficient  $A_1$  in a similar fashion to the one used for the sinh-Gordon model earlier [86]

$$A_1 = \pm\sqrt{2} \cosh a_1\pi, \quad a_1 \in \mathbb{R} \quad (2.52)$$

the equation (2.49) is solved by

$$g = e^{\Phi_0} = \left( \frac{1}{2} + \cosh \frac{2\pi a_1}{3} \right)^{-1}.$$

Comparing this with the sinh solution (2.40) one can express  $E_0 = e^{-z_0}$  in terms of the boundary

$$e^{-\Phi_0} = 1 + \frac{3/2}{\sinh^2 \frac{z_0}{2}} = \frac{1}{2} + \cosh \frac{2\pi a_1}{3}. \quad (2.53)$$

The relevant solution for  $A_1 = \sqrt{2} \cosh a_1\pi$  is

$$E = e^{-z_0} = \frac{-\sqrt{3} + u + u^{-1}}{\sqrt{3} + u + u^{-1}}$$

with  $u = e^{a_1\pi/3}$ . Now all variables in (2.48) depend on  $a_1$ . The relation (2.53) can also be used to test whether  $z_0$  can always be chosen to be positive. The equation implies

$$\frac{3}{2} \left( \cosh \frac{2\pi a_1}{3} - \frac{1}{2} \right) + 1 = \coth^2 \frac{z_0}{2}.$$

The positive root indeed allows one to choose a positive  $z_0$  for any  $a_1$ .

The ratio  $\frac{a^*}{a}$  (2.48) can now be rewritten as

$$\begin{aligned}\frac{a^*}{a} &= \frac{1 + i2\sqrt{3}\lambda u + u^2 - 4\lambda^2 u^2 + i2\sqrt{3}\lambda u^3 + u^4}{1 - i2\sqrt{3}\lambda u + u^2 - 4\lambda^2 u^2 - i2\sqrt{3}\lambda u^3 + u^4} \\ &= \frac{-\lambda^2 + i\frac{\sqrt{3}}{2}\lambda(u^{-1} + u) + \frac{1}{4}(u^2 + u^{-2} + 1)}{-\lambda^2 - i\frac{\sqrt{3}}{2}\lambda(u^{-1} + u) + \frac{1}{4}(u^2 + u^{-2} + 1)}.\end{aligned}$$

This expression can be factorised with  $x = 2 + ia_1$  and  $y = \bar{x}$

$$\frac{a^*}{a} = \frac{\left(i\lambda + \sin \frac{\pi x}{6}\right)\left(i\lambda + \sin \frac{\pi y}{6}\right)}{\left(i\lambda - \sin \frac{\pi x}{6}\right)\left(i\lambda - \sin \frac{\pi y}{6}\right)}.$$

Using the block notation (1.29) and observing the fact that

$$1/2\left(i\lambda \pm \sin \frac{\pi x}{6}\right) = \mp \sinh\left(\frac{\theta}{2} \mp \frac{i\pi x}{12}\right) \sinh\left(\frac{\theta}{2} - \frac{i\pi x}{12}(6 \pm x)\right)$$

one can rewrite  $\frac{a^*}{a}$  as

$$\frac{a^*}{a} = \frac{(-y)(x-6)}{(x)(6-y)}.$$

Therefore the entire reflection factor (2.46) is given by

$$K_+^> = (1)(3)^2(5)\frac{(2ia_1 - 2)(2ia_1 - 4)}{(2ia_1 + 2)(2ia_1 + 4)}.$$

For the case  $A_1 = -\sqrt{2} \cosh a_1 \pi$  the relevant solution to (2.53) is

$$e^{-z_0} = \frac{\sqrt{3} + u + u^{-1}}{-\sqrt{3} + u + u^{-1}}.$$

This yields the reflection factor

$$K_-^> = (1)(3)^2(5)\frac{(2ia_1 + 2)(2ia_1 + 4)}{(2ia_1 - 2)(2ia_1 - 4)}.$$

For the boundary condition  $A_1 = \pm\sqrt{2}$  the reflection factors are

$$K_-^> = (1)(3)^2(5)(2)^2(4)^2$$

$$K_-^> = (1)(3)^2(5)\frac{1}{(2)^2(4)^2}.$$

So far only the regime  $A_1^2 > 2$  has been investigated. For the remaining range of  $A_1$  a suitable parametrisation is

$$A_1 = \pm\sqrt{2} \cos a_1 \pi \quad a_i \in [0, 1/2] \text{ or } a_i \in [-1/2, 0].$$

Notice that this is same parametrisation as before, only with the change  $ia_1 \rightarrow a_1$ . The change of  $u$  as defined above from a real to a complex expression does not change the outcome. So, the reflection coefficients are

$$K_{\pm}^{\leq} = (1)(3)^2(5) \left\{ \frac{(2a_1 - 2)(2a_1 - 4)}{(2a_1 + 2)(2a_1 + 4)} \right\}^{\pm 1}.$$

They coincide with the expressions of  $K_{\pm}^{\geq}$  for  $A_1 = \pm\sqrt{2}$  and, for the natural Neumann condition  $A_1 = 0, a_1 = \pm\frac{1}{2}$ , they are unity.

One should still check whether  $z_0$  can be chosen to be positive. With the new parametrisation (2.50) is solved by

$$g = \left( \frac{1}{2} + \cos \frac{2\pi a_1}{3} \right)^{-1} = e^{\Phi_0}.$$

Due to the symmetry of (2.50) under  $a_1 \rightarrow a_1 + 1$  and  $a_1 \rightarrow a_1 + 2$  there are two more solutions differing from the above in the shift of  $a_1$ . They will be neglected in the following. For all  $a_1 \in [-\frac{1}{2}, \frac{1}{2}]$ ,  $g$  is greater than one thus indicating the relevant solution is the sinh solution. So, to match the boundary conditions,  $z_0$  has to satisfy

$$\frac{1}{2} + \cos \frac{2\pi a_1}{3} = e^{-\Phi_0} = 1 + \frac{3/2}{\sinh^2 \frac{z_0}{2}}.$$

Implying

$$\frac{3}{2} \left( \cos \frac{2\pi a_1}{3} - \frac{1}{2} \right) + 1 = \coth^2 \frac{z_0}{2}.$$

So that the positive root always allows one to pick a positive  $z_0$  for any  $a_1 \in [-\frac{1}{2}, \frac{1}{2}]$ .

It possible to write all this in one formula

$$A_1 = \pm\sqrt{2} \cos a_1 \pi, \text{ with either } a_1 = \pm b_1, b_1 \in [0, \frac{1}{2}] \text{ or } a_1 \in i\mathbb{R} \quad (2.55)$$

$$K_{\pm} = (1)(3)^2(5) \left\{ \frac{(2a_1 - 2)(2a_1 - 4)}{(2a_1 + 2)(2a_1 + 4)} \right\}^{\pm 1}.$$

Case B:  $A_1^2 = 2$

In case B the equation (2.42) simplifies to

$$-A_1 A_0 e^{\Phi_0/2} - \frac{16 - A_0^2}{4} e^{-\Phi_0} + 6 = 0$$

which can be written as a cubic polynomial in  $g = e^{\Phi_0/2}$

$$f(g) = A_1 A_0 g^3 - 6g^2 + \frac{16 - A_0^2}{4} = 0. \quad (2.56)$$

Again a discussion of this polynomial  $f(g)$  is going to tell which solution is to be used with what boundary condition. The polynomial  $f(g)$  has two turning points  $g_1 = 0$  and  $g_2 = \frac{4}{A_0 A_1}$ , where  $g_1$  is a local maximum and  $g_2$  is a local minimum. The values of the polynomial at these points are  $f(g_1) = \frac{16 - A_0^2}{4}$  and  $f(g_2) = -\frac{(A_0 - 8)^2}{4A_0^2} < 0$ . The last interesting point of the polynomial for the discussion is  $g = 1$ ,  $f(1) = -\frac{1}{4}(A_0 - 2A_1)^2 < 0$ .

Depending on the sign of the coefficient of  $p^3$  in (2.56) there are two cases.

- $A_1 A_0 < 0$ : In this case  $g_2$  is negative, so there can be only one zero to the right of  $g_1 = 0$  because  $f(g)$  tends to  $-\infty$  for  $g \rightarrow \infty$ . The zero only exist if  $f(0) = \frac{16 - A_0^2}{4} > 0$  i.e.  $A_0^2 < 16$ . Because  $f(1)$  is negative the zero has to occur between  $0 < p = e^{\Phi_0/2} < 1$ . This indicates that the sinh solution is relevant.

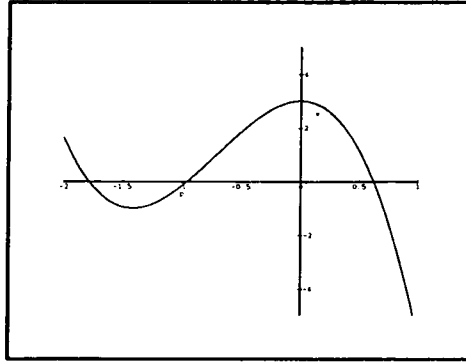


Fig. 2.10:  $f(p)$  for  $A_0 = 2$  and  $A_1 = -\sqrt{2}$

There is no zero with a positive  $p$  for  $A_0^2 > 16$ . Figure illustrates this with  $A_0 = 2$ ,  $A_1 = -\sqrt{2}$ .

- $A_1 A_0 > 0$ : Here  $g_2$ , the position of the local minimum, is positive. If  $A_0^2 < 16$  the polynomial is positive for  $g = 0$ ,  $f(0) > 0$ . Also,  $f(g)$  tends to  $+\infty$  as  $g \rightarrow +\infty$ .

Thus there are two positive zeros. One is always between 0 and 1 and the other one always greater than 1 because  $f(1)$  is negative. Therefore both the sinh and the cosh solution are relevant.

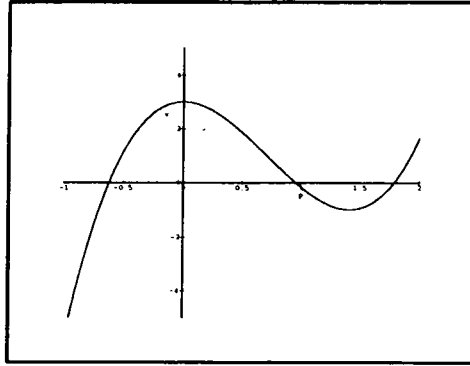


Fig. 2.11:  $f(p)$  with  $A_0 = 2$ ,  $A_1 = \sqrt{2}$

This is illustrated in figure (2.11) with  $A_0 = 2$ ,  $A_1 = \sqrt{2}$ . If  $A_0^2 > 16$  the polynomial is negative at  $p = 0$  and  $p = 1$ , therefore there is only one zero for which  $p$  is greater than 1.

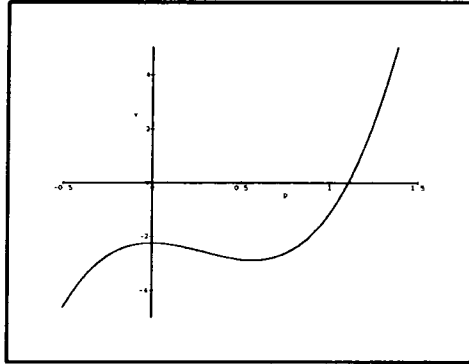


Fig. 2.12:  $f(p)$  for  $A_0 = 5$ ,  $A_1 = \sqrt{2}$

Thus only the cosh solution is relevant. This situation is sketched in figure (2.12) with  $A_0 = 5$ ,  $A_1 = \sqrt{2}$ .

First the cases with  $A_0^2 < 16$  shall be investigated. A natural parametrisation of  $A_0$  seems to be

$$A_0 = 4 \cos a_0 \pi \quad a_0 \in \mathbb{R}. \quad (2.57)$$

With  $A_1 = \pm\sqrt{2}$ ,  $s = e^{ia_0\pi/3}$  and  $q = g\sqrt{2} = \sqrt{2}e^{\Phi_0/2}$  the equation (2.56) is transformed to

$$\pm(s^3 + s^{-3})q^3 - 3q^2 - (s^3 - s^{-3})^2 = 0 \quad (2.58)$$

where the coefficient of  $p^3$  in (2.58) depends on the sign of  $A_1$ . Due to its symmetry under  $s \rightarrow \Omega s \rightarrow \Omega^2 s$  the three solutions to the cubic equation (2.58) are

$$q_j^\pm = \pm \frac{s^3 - s^{-3}}{\Omega^j s^2 - \Omega^{-j} s^{-2}} \quad (2.59)$$

where  $\Omega = e^{i2\pi/3}$ ,  $j \in \{0, 1, 2\}$ . Introducing the variable  $S_j = \Omega^{j/2} s = e^{i\frac{\pi}{3}(a_0+j)}$  the solution can be written as

$$q_j^\pm = \pm (-1)^j \frac{S_j^3 - S_j^{-3}}{S_j^2 - S_j^{-2}} = \pm (-1)^j \frac{\sin(a_0 + j)\pi}{\sin \frac{2\pi}{3}(a_0 + j)}.$$

Also, note that the boundary condition (2.57) can be written as

$$A_0 = 4(-1)^j \cos(a_0 + j)\pi \quad a_0 \in \mathbb{R}, \quad j \in \mathbb{Z}.$$

All boundary conditions can be matched by the sinh or cosh solutions, i.e. a positive  $z_0$  can be chosen. For instance for the case  $A_1 = \sqrt{2}$  the sinh solution has to satisfy

$$e^{-\Phi_0} = 1 + \frac{3}{2}(\coth^2 \frac{z_0}{2} - 1) = \frac{2}{q^2} = 2 \frac{\sin^2 \frac{2\pi a_0}{3}}{\sin^2 \pi a_0}. \quad (2.60)$$

This means the positive square root of

$$\left(\frac{2}{q^2} - 1\right)^2 + 1 = \coth^2 \frac{z_0}{2}$$

will always allow one to choose a positive  $z_0$  such that (2.60) is fulfilled because  $\frac{2}{q^2} > 1$  due to the choice of  $a_0$ .

As in case A the calculation will differ a little depending on whether  $A_1$  is positive or negative. First set  $A_1 = +\sqrt{2}$  and write  $q_j^+ = q_j$ . To express  $E = e^{-z_0}$  in terms of  $a_0$  for the sinh solution the equation

$$\frac{q_j}{\sqrt{2}} = e^{\Phi_0/2} = \frac{1 - E_0^j}{\sqrt{1 + 4E_0^j + (E_0^j)^2}} \quad (2.61)$$

has to be solved for  $E_0^j$  where the index  $j$  indicates to what solution the  $E$  belongs. One of the solutions is

$$E_0^j = (\sqrt{3} - 2) \frac{S_j^2 + S_j^{-2} - \sqrt{3}}{S_j^2 + S_j^{-2} + \sqrt{3}} = (\sqrt{3} - 2) \frac{2 \cos \frac{2\pi}{3}(a_0 + j) - \sqrt{3}}{2 \cos \frac{2\pi}{3}(a_0 + j) + \sqrt{3}}. \quad (2.62)$$

If one wants to work with the cosh solution instead one has to change the sign of  $E_0^j$  in (2.62). Because the sign also has to be changed in (2.48) these changes cancel and further calculation reveals the same results for both solutions. Using (2.62) to write all variables in terms of  $S_j$ , i.e. the boundary condition (2.48) simplifies to

$$\frac{a^*}{a} = - \left( i\lambda qp + p'q - pq' + \frac{1}{2\sqrt{3}} \left[ \left( \frac{S_j^3 + S_j^{-3}}{S_j^2 + S_j^{-2}} \right)^2 + \frac{(S_j^3 + S_j^{-3})(S_j^2 - S_j^{-2})}{S_j^3 - S_j^{-3}} \right] pq \right) / c.c. \quad (2.63)$$

This expression factorises in similar fashion to case A as follows

$$\begin{aligned} \frac{a^*}{a} &= - \frac{\left( i\lambda + \frac{\sqrt{3}}{2} \right) \left( (i\lambda)^2 + i\lambda \frac{\sqrt{3}}{2} (S_j^2 + S_j^{-2}) + \frac{1}{4} (S_j^4 + 1 + S_j^{-4}) \right)}{\left( -i\lambda + \frac{\sqrt{3}}{2} \right) \left( (i\lambda)^2 - i\lambda \frac{\sqrt{3}}{2} (S_j^2 + S_j^{-2}) + \frac{1}{4} (S_j^4 + 1 + S_j^{-4}) \right)} \\ &= - \frac{1}{(2)(4)} \frac{(4a_0 + 4j - 2)(4a_0 + 4j - 4)}{(4a_0 + 4j + 2)(4a_0 + 4j + 4)}. \end{aligned}$$

In the last line the block notation (1.29) was used. For each  $j$  the equations (2.59) and (2.61) impose limits on  $a_0$ . For the cosh solution one needs  $q_j^2/2 > 1$  and for the sinh solution  $0 < q_j^2/2 < 1$ . Also, because of (2.61),  $0 < E_0^j < 1$ . Because the sinh and the cosh solution give the same reflection factor there is no need to distinguish between them and the allowed values are  $a_0 \in [-1, 1]$  for  $j = 0$ ,  $a_0 \in [1, 3]$  for  $j = 1$  and  $a_0 \in [3, 5]$  for  $j = 2$ .

The entire reflection factor is, for  $A_1 = \sqrt{2}$  and  $A_0 = (-1)^j \cos(a_0 + j)\pi$  with  $a_0 = b_0 + j$   $b_0 \in [-1, 1]$ ,

$$K^+(a_0, j) = -(1)(3)^2(5) \frac{1}{(2)(4)} \frac{(4a_0 + 4j - 2)(4a_0 + 4j - 4)}{(4a_0 + 4j + 2)(4a_0 + 4j + 4)}.$$

It is quite instructive to evaluate the expression for some special values. For instance  $j = 0$  and

- $A_0 = 0$  i.e.  $a_0 = 1/2$  gives the factor

$$K^+(a_0, 0) = (1)(3)^2(5) \frac{1}{(2)^2(4)^2}.$$

This coincides with the expression for the case A and shows that both cases are continuously connected.



- $A_0 = 2\sqrt{2}$  i.e.  $a_0 = 1/4$  gives the factor

$$K^+(a_0, 0) = -\frac{1}{(2)(4)}.$$

In this case an simple calculation allows one to check this result because the solution  $\Phi_0(x, t) = 0$  is permitted with these special boundary conditions.

- $A_0 = 4$  i.e.  $a_0 = 0$  gives the factor

$$K^+(a_0, 0) = -(1)(3)^2(5)\frac{1}{(2)^3(4)^3}.$$

So this looks extremely similar to the first special case and it seems that the second parameter of the boundary conditions amplifies the other one.

For  $A_1 = -\sqrt{2}$  one has to make use of the second solution to (2.61) which is simply the inverse to the one used previously

$$E_0^j = \frac{1}{(\sqrt{3} - 2)} \frac{S_j^2 + S_j^{-2} + \sqrt{3}}{S_j^2 + S_j^{-2} - \sqrt{3}}.$$

Also, one has to use the  $q_j^-$  solutions. With these changes the result for the ratio (2.48) is

$$K^-(a_0, j) = -(1)(3)^2(5)(2)(4) \frac{(4a_0 + 4j + 2)(4 + 4a_0 + 4j)}{(4a_0 + 4j - 2)(4a_0 + 4j - 4)}$$

The allowed values for  $a_0$  are  $a_0 = b_0 + j$  where  $b_0 \in [\frac{1}{4}, 1\frac{1}{4}] \cup [1\frac{3}{4}, 2\frac{3}{4}]$ .

- For  $A_0 = -2\sqrt{2}$  i.e.  $a_0 = 1\frac{1}{4}$  one gets the factor

$$K^- = -(2)(4)$$

which can again be verified by a simple calculation.

- Also,  $A_0 = -4$  i.e.  $a_0 = 1$  give the same as before

$$K^- = (1)(3)^2(5).$$

- And  $A_0 = 0$  i.e.  $a_0 = \frac{1}{2}$  gives

$$K^- = (1)(3)^2(5)(2)^2(4)^2$$

which is again the result found in case A.

The reflection factors are periodic under  $a_0 \rightarrow a_0 + 3$ . They seem to inherit their periodicity from the solution of (2.56). A change from  $j$  to  $j + 1$  effectively inverts the  $a_0$  dependent part of the reflection factor. The factor

$$-\left(\frac{1}{(2)(4)}\right)^{\pm 1}$$

could be interpreted as caused by the  $A_1 = \pm\sqrt{2}$  parameter. All factors are symmetric under  $a_0 \rightarrow -a_0$ .

*Reflection factors for  $A_0^2 > 16$  and  $A_0 A_1 > 0$*

The obvious parametrisation of this case is

$$A_0 = \pm \cosh a_0 \pi \quad a_0 \in \mathbb{R}.$$

The real solution to (2.58) is then

$$q = \frac{\sinh a_0 \pi}{\sinh \frac{2\pi}{3} a_0}.$$

From the discussion of the polynomial one knows that only the cosh solution is relevant. There are two cases to look at:

a)  $A_0 > 4$  and  $A_1 = -\sqrt{2}$ . Here the cosh solution is allowed for all  $a_0$  because

$$e^{\Phi_0/2} = \frac{1}{\sqrt{2}} \frac{\sinh a_0 \pi}{\sinh \frac{2\pi}{3} a_0} > 1 \quad \forall a_0 \in \mathbb{R}$$

and also  $E_0$  which is

$$E_0 = -(\sqrt{3} - 2) \frac{2 \cosh \frac{2\pi}{3} a_0 - \sqrt{3}}{2 \cosh \frac{2\pi}{3} a_0 + \sqrt{3}}$$

is positive and smaller than  $2 - \sqrt{3}$  for all  $a_0$ . Therefore the reflection factor is

$$K(a_0) = -(1)(3)^2(5) \frac{1}{(2)(4)} \frac{(4ia_0 - 2)(4ia_0 - 4)}{(4ia_0 + 2)(4ia_0 + 4)}.$$

b)  $A_0 < -4$  and  $A_1 = \sqrt{2}$ . In this the same solution for (2.58) can be used as in case b). And

$$E_0 = -[(\sqrt{3} - 2) \frac{2 \cosh \frac{2\pi}{3} a_0 - \sqrt{3}}{2 \cosh \frac{2\pi}{3} a_0 + \sqrt{3}}]^{-1}$$

is greater than  $2 + \sqrt{3}$  for all  $a_0 \in \mathbb{R}$ . And therefore the reflection factor is

$$K(a_0) = -(1)(3)^2(5)(2)(4) \left\{ \frac{(4ia_0 - 2)(4ia_0 - 4)}{(4ia_0 + 2)(4ia_0 + 4)} \right\}^{-1}.$$

#### 2.4.2 The energy of the static background solutions in dependence on the boundary

For the static background solution (2.40) one can work out the energy in the following way. The energy density for the Toda theory on the half line splits into two parts, the kinetic and potential energy density on the negative axis

$$\mathcal{E} = \frac{1}{2}[(\dot{\Phi}_0)^2 + (\Phi'_0)^2] + [e^{2\Phi(x,t)} + 2e^{-\Phi(x,t)} - 3].$$

The second part is the contribution of the boundary potential

$$\mathcal{B} = A_1 e^{\Phi_0(x)} + A_0 e^{-\Phi_0(x)/2}.$$

As  $\Phi_0(x)$  does not depend on time, the expression for the total energy  $W$  simplifies to

$$\begin{aligned} W &= \left[ \int_{-\infty}^0 dx \frac{1}{2} (\Phi'_0(x))^2 + e^{2\Phi_0(x)} + 2e^{-\Phi_0(x)} - 3 \right] \\ &\quad + A_1 e^{\Phi_0} + A_0 e^{-\Phi_0/2} \\ &= \left[ \int_{-\infty}^0 dx \left( \frac{\Phi'_0(x)}{\sqrt{2}} + (e^{\Phi_0(x)} - 1) \sqrt{2e^{-\Phi_0(x)} + 1} \right)^2 \right. \\ &\quad \left. - \sqrt{2} \Phi'_0(x) (e^{\Phi_0(x)} - 1) \sqrt{2e^{-\Phi_0(x)} + 1} \right] + A_1 e^{\Phi_0} + A_0 e^{-\Phi_0/2} \end{aligned}$$

Using the Bogomolny argument one can now estimate that the energy is

$$W \geq -\sqrt{2} \left[ \int_{-\infty}^0 dx \Phi'_0(x) (e^{\Phi_0(x)} - 1) \sqrt{2e^{-\Phi_0(x)} + 1} \right] + A_1 e^{\Phi_0} + A_0 e^{-\Phi_0/2}.$$

Using the substitution rule twice, the second time for  $e^{-z} = \frac{\sinh^2 u}{2}$ , and  $\sinh^2 u_0 = 2$   $\sinh^2 u_1 = 2e^{\Phi_0}$  this can be written as

$$\begin{aligned} W &\geq -\sqrt{2} \left[ \int_0^{\Phi(0)} dz (e^z - 1) \sqrt{2e^{-z} + 1} \right] + A_1 e^{\Phi_0} + A_0 e^{-\Phi_0/2} \\ &= 2\sqrt{2} \left[ \int_{u_0}^{u_1} du \cosh^2 u \frac{2 - \cosh^2 u}{\sinh^2 u} \right] + A_1 e^{\Phi_0} + A_0 e^{-\Phi_0/2} \\ &= 2\sqrt{2} \left[ \cosh u \frac{\cosh^2 u}{\sinh^2 u} \right]_{u_0}^{u_1} + A_1 e^{\Phi_0} + A_0 e^{-\Phi_0/2} \\ &= -3\sqrt{6} + \sqrt{2} e^{\Phi_0} (1 + 2e^{-\Phi_0})^{3/2} + A_1 e^{\Phi_0} + A_0 e^{-\Phi_0/2}. \end{aligned} \tag{2.64}$$

For the case A  $A_1^2 < 2$  the energy of the sinh solution is

$$W \geq -3\sqrt{6} + \frac{2\sqrt{2}}{2 + \cos \frac{\pi a_1}{3}} (8 \cos^3 \frac{\pi a_1}{3} \pm \cos \pi a_1).$$

Which is bounded below and indicates that the solution is stable [92]. For  $A_1^2 < 2$  one has to replace  $\cos$  with  $\cosh$ .

For case B the result for the sinh solution is

$$W \geq -3\sqrt{6} + \left\{ \left( \sin^2 \pi a_0 + 4 \sin^2 \frac{2\pi a_0}{3} \right)^{3/2} \pm \sin \frac{2\pi a_0}{3} \sin^2 \pi a_0 \right. \\ \left. + 4 \cos \pi a_0 \sin \frac{2\pi a_0}{3} \sin^2 \pi a_0 \right\} / \left\{ \sqrt{2} \sin^2 \frac{2\pi a_0}{3} \sin \pi a_0 \right\}$$

which has no singularity and is bounded below. Again this indicates that the solution is stable [92].

## 2.5 Conclusions

In both cases A and B classical reflection factors were found. They can be written in the block notation (1.29) of the the  $S$ -matrix. All factors fulfilled the classical reflection bootstrap equation (2.12) with  $\theta = i\pi/3$  corresponding to a pole of the  $S$ -matrix. One can write the reflection factors of both cases in one formula. For  $j = 1$  this is

$$A_0 = 4 \cos a_0 \pi \quad a_0 \in \begin{cases} i\mathbb{R} & \text{for } A_0^2 > 16 \\ [-1, 1] & \text{for } A_1 > 0 \\ a_0 \in [\frac{1}{4}, 1\frac{1}{4}] \cup [1\frac{3}{4}, 2\frac{3}{4}] & \text{for } A_1 < 0 \end{cases} \\ A_1 = \pm \sqrt{2} \cos a_1 \pi \quad a_1 \in i\mathbb{R} \text{ or } a_1 = \pm b_1 \quad b_1 \in [0, 1/2] \quad (2.65)$$

$$K = -(1)(3)^2(5) \left\{ (2)(4) \frac{(4a_0 - 2)(4a_0 - 4)(2a_1 - 2)(2a_1 - 4)}{(4a_0 + 2)(4a_0 + 4)(2a_1 + 2)(2a_1 + 4)} \right\}^{\pm 1}.$$

To prove that the classical reflection bootstrap (2.7) is fulfilled one has to show that

$$K(\theta) = K(\theta + i\pi/3)K(\theta - i\pi/3).$$

The parts of the reflection factor not depending on  $a_0$  and  $a_1$  have to fulfil the bootstrap separately. Due to (1.30) and the Coxeter number being  $h = 6$

$$(x)_{\theta+i\pi/3}(x)_{\theta-i\pi/3} = (x+2)_{\theta}(x-2)_{\theta}.$$

Which implies for the constant factors

$$\begin{aligned} K(\theta+i\pi/3)K(\theta-i\pi/3) &= +(-1)(3)(1)^2(5)^2(3)(7)\left((0)(4)(2)(6)\right)^{\pm 1} \\ &= -(1)(3)^2(5)\left((2)(4)\right)^{\pm 1} = K(\theta). \end{aligned} \quad (2.66)$$

Similarly the part depending on  $a_0$  gives

$$\begin{aligned} K(\theta+i\pi/3)K(\theta-i\pi/3) &= \\ &\left\{ \frac{(4a_0-4)(4a_0)}{(4a_0+4)(4a_0)} \frac{(4a_0-6)(4a_0-2)}{(4a_0+2)(4a_0+6)} \frac{(2a_1-4)(2a_1)}{(2a_1)(2a_1+4)} \right. \\ &\quad \left. \frac{(2a_1-6)(2a_1-2)}{(2a_1+2)(2a_1+6)} \right\}^{\pm 1} \\ &= \left\{ \frac{(4a_0-2)(4a_0-4)}{(4a_0+2)(4a_0+4)} \frac{(2a_1-2)(2a_1-4)}{(2a_1+2)(2a_1+4)} \right\}^{\pm 1} = K(\theta). \end{aligned} \quad (2.67)$$

Together (2.66) and (2.67) show that the bootstrap is fulfilled. One should note that the two parts of the  $a_0$  and  $a_1$  independent part of (2.65) namely  $(1)(3)^2(5)$  and  $-(2)(4)$  obey the reflection bootstrap (2.12) by themselves. Because the factors obey the classical reflection bootstrap (2.12) any multiple of them with a reflection matrix obeying the reflection bootstrap equation (2.12) solves this equation (2.7) again. Both cases A and B are continuously connected. Case A can also correspond to the natural Neumann condition. However, in that case, the reflection factor is one. It is worth mentioning that the expression (2.65) looks similar to the one for the sinh-Gordon model (2.32) with the difference that one has two parameters in the Bullough-Dodd model. Also, one should not overlook that the Bullough-Dodd model inherits the reflection factor of the  $a_2^{(1)}$  model (2.28) which is the the same as the constant factor (2.66).

Kim [83] gives an “exact” solution to (2.7) for the  $a_2^{(2)}$  theory

$$K(\theta) = [1/2][3/2] \sqrt{\frac{[1]}{[2]}}$$

where  $h = 3$ . This clearly fulfils the bootstrap but has the serious drawback of introducing square root branch cuts in  $\theta$  which are hard to explain.

## Chapter III

### Breathers in Affine Toda Field Theory

#### 3.1 Introduction

The sine-Gordon model is the simplest example of the  $a_n^{(1)}$  affine Toda field theories, it corresponds to  $a_1^{(1)}$ . An interesting feature of the sine-Gordon model is the existence, not only of soliton and anti-soliton solutions, but of oscillating, solitonic solutions, breathers. A breather is the bound state of a soliton and an anti-soliton in the sine-Gordon model. Because  $a_n^{(1)}$  affine Toda field theory is the generalisation of the sine-Gordon theory it is legitimate to ask whether breather solutions exist there as well. There have been many speculations [35,45,67,96] about their existence. Also calculations of scattering processes [41] in  $a_n^{(1)}$  affine Toda solitons give hints of the existence of breathers. The soliton  $S$ -matrices have poles which should correspond to bound states of soliton pairs. In [95] an explicit construction of breather solutions was given. This construction and the discussion of their topological charges will be described in the following sections.

For the  $a_1^{(1)}$  Toda field theory, which is the sine-Gordon theory, the single soliton solution is given by,

$$\phi = \frac{i\sqrt{2}}{\beta} \ln \left( \frac{1 - e^{\sigma(x-vt)+\rho}}{1 + e^{\sigma(x-vt)+\rho}} \right), \quad (3.1)$$

with the constraint  $\sigma^2(1 - v^2) = 4m^2$ . The parameter  $\rho = \eta + i\xi$  is complex, and its imaginary part determines the topological charge of the soliton. For  $\xi = \pm \frac{\pi}{2}$  the expression (3.1) becomes real and is either a soliton or anti-soliton. They have the same mass, equal to  $\frac{8m}{\beta^2}$ . According to [97] the first time breathers have been mentioned is in the work of Kochendörfer and Seeger [98]. They were investigating the Frenkel-Kontorova model which describes a one dimensional line of atoms which are coupled to their nearest neighbours and interact with sine-like background potential.

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This chapter is a slightly modified version of the publication [95]. Section 3.3 to 3.5 have been taken from [95] with only minute changes.

In the continuum limit the displacement of the atoms is given by solutions of the sine-Gordon equation, this is discussed in [98]. In part three of their work they describe static “dislocations” and their interactions with breathers, which they call oscillatory displacements. They derive their solutions from Bäcklund transformations. In their interpretation, breathers correspond to solutions which travel faster than the speed of sound in the line of atoms. They also mention the double-sine-Gordon equation [99] and that earlier work has shown that it allows soliton solutions for certain parameters [100]. Unfortunately they give no information to show what this equation is good for in the context of solid state physics. They also mention a two-dimensional formulation of the sine-Gordon equation.

The sine-Gordon breather is constructed from two approaching soliton solutions by changing the velocity  $v$  into  $iv$ ,

$$\phi_{\text{breather}} = \frac{i\sqrt{2}}{\beta} \ln \left( \frac{1 - e^{\sigma(x-ivt)+\rho_1} - e^{\sigma(x+ivt)+\rho_2} - v^2 e^{2\sigma x+(\rho_1+\rho_2)}}{1 + e^{\sigma(x-ivt)+\rho_1} + e^{\sigma(x+ivt)+\rho_2} - v^2 e^{2\sigma x+(\rho_1+\rho_2)}} \right).$$

Taking  $\xi_1 = -\xi_2 = -\frac{\pi}{2}$  and  $\eta_1 = \eta_2 = \eta$ , yields a soliton solution oscillating about the point  $\frac{\ln(v^2)+2\eta}{2\sigma}$ . As it is constructed from a soliton-anti-soliton pair, this sine-Gordon breather has zero topological charge; its mass is equal to  $\frac{16m}{\beta^2\sqrt{1+v^2}}$  and hence smaller than for a two-soliton solutions. Following the prescription of taking imaginary velocity, the breathers of the  $a_n^{(1)}$  affine Toda theories can be constructed from two soliton solutions. It turns out that, in order for the energy of the breathers built from two solitons to be real, the constituent solitons must be of the same mass and approaching each other with the same imaginary velocity, resulting in a stationary breather. To obtain a moving breather, one can apply the usual Lorentz boost to the breather solution. The condition of real energy also produces an expression for the masses of these breathers which is less than the sum of the constituent solitons. One type of breathers can carry topological charge which coincides with the topological charge of a certain single soliton, while the other type has zero topological charge. The work presented here has been extended for other algebras in [101], one can find results for breathers in Toda theories other than those corresponding to  $a_n^{(1)}$ .

### 3.2 Breather Solutions for $a_n^{(1)}$ Toda theory

Following the prescription used to obtain breather solutions in the Sine-Gordon model



one can change the velocity  $u$  into an imaginary  $iv$  in the  $\tau$ -functions of the two-soliton solution of  $a_n^{(1)}$  affine Toda soliton solutions.

However one has to be careful with this analytic continuation  $u \rightarrow iv$  as one wants to keep the energy and the momentum real although the densities become in general complex.

Changing  $u$  into  $iv$  also means changing a real rapidity into an imaginary rapidity, with a relation between velocity  $v$  and rapidity  $\Theta$ ,  $v = \tan(-i\Theta)$ . From the light-cone energy-momentum (1.48) of the two soliton solution, it is clear that a real energy and momentum can be achieved provided that the two solitons forming a breather are of the same mass and moving towards each other with the same velocity giving a stationary breather. One can make an oscillating solution from solitons of two different masses, but the energy and momentum of this solution will not be real. Generally, one can add a real rapidity  $\Theta_0$  as a phase in the energy-momentum tensor, which acts as a Lorentz boost to the breather solution. Thus,

$$P_{breather}^{\pm} = \frac{4hm_a}{\sqrt{2}\beta^2} \cos(\Theta_a) e^{\mp\Theta_0}, \quad (3.2)$$

$m_a$  above is the mass of the classical Toda particle of the  $a_n^{(1)}$  theory. For simplicity, in what follows only stationary breathers are considered. Hence (3.2) becomes,

$$P_{breather}^{\pm} = \frac{4h}{\sqrt{2}\beta^2} \frac{m_a}{\sqrt{1+v^2}}, \quad (3.3)$$

and the mass of a breather is,

$$M_{breather} = \frac{2M_a}{\sqrt{1+v^2}} = \frac{4hm_a}{\beta^2\sqrt{1+v^2}}. \quad (3.4)$$

Obviously the mass of a breather is less than the sum of its constituent solitons. This result generalises the sine-Gordon case, i.e. taking  $h = 2$  in (3.4) gives the mass of the sine-Gordon breather.

The masses  $m_a$  of the fundamental particles of the  $a_n^{(1)}$  series are degenerate with respect to the  $Z_2$  symmetry of the  $A_n$  Dynkin diagram, i.e.  $m_a = m_{h-a}$ . Hence, there are two possibilities of forming a breather. Either the two constituent solitons are of the same species, these breathers will be called *type A* breathers, or the two constituent solitons are of opposite species, *type B* breathers. Exceptions to this

classification are the breathers constructed from solitons of species  $(n + 1)$  of the  $a_{2n+1}^{(1)}$  theories. These breathers are sine-Gordon embedded breathers which belong to both type A and B as will be explained in the following sections.

Looking back at the  $\tau$ -function of a two soliton solution (1.44) of the same constituent mass, choosing  $u_a = -u_b = iv$  yields the breather  $\tau$ -function,

$$\begin{aligned} \tau_j^{(ab)} = & 1 + \exp[\sigma_a(x - ivt) + \rho_a + ij\theta_a] + \exp[\sigma_b(x + ivt) + \rho_b + ij\theta_b] \\ & + \exp[\sigma_+x + \lambda + \rho_+ + ij\theta_+], \end{aligned} \quad (3.5)$$

the interaction coefficient (1.45) is written as  $A = e^\lambda$  with  $\lambda = \zeta + i\delta$ , where  $\zeta, \delta \in \mathbb{R}$  and

$$\begin{aligned} \sigma_\pm &= \sigma_a \pm \sigma_b, & \rho_\pm &= \rho_a \pm \rho_b, & \theta_\pm &= \theta_a \pm \theta_b, \\ \eta_\pm &= \eta_a \pm \eta_b, & \xi_\pm &= \xi_a \pm \xi_b. \end{aligned}$$

The positive interaction coefficient has  $\delta = 0$  and the negative one  $\delta = \pi$ . Note that for solitons of the same mass,  $\sigma_a = \sigma_b$ . The ansatz (1.38) requires each  $\alpha_j$  component of the solution  $\phi$  to be well defined in order to have a well defined solution. Thus, for each  $j$ , the ratio  $\frac{T_j}{\tau_0}$  must not become zero or infinite. Evaluation of the behaviour of the  $\tau$ -function can be done easily by writing the real and imaginary part of (3.5) explicitly. It turns out that to avoid the real and imaginary part of (3.5) becoming zero simultaneously at the same point, the parameters  $\xi_+$  and  $\eta_-$  are restricted to a certain range of definition. For later use one should write down the breather  $\tau$  function for positive and negative interaction coefficient. For both cases define  $R(x) = \frac{1}{2}(2\sigma x + \zeta + \eta_+)$  and  $T^\pm = \frac{1}{2}(\xi_\pm + j\theta_\pm)$ . In the case of the positive interaction coefficient the  $\tau$  function is then

$$\begin{aligned} \tau_j^{ab} = & 2 \exp \left( R(x) + T_j^+ + i\frac{\delta}{2} \right) \left\{ \cosh R(x) \cos T_j^+ + e^{\frac{-\zeta}{2}} \cosh \frac{\eta_-}{2} \cos(\sigma vt - T_j^-) \right. \\ & \left. + i \left( \sinh R(x) \sin T_j^+ - e^{\frac{-\zeta}{2}} \sinh \frac{\eta_-}{2} \sin(\sigma vt - T_j^-) \right) \right\}. \end{aligned} \quad (3.6)$$

For the negative interaction coefficient the result is

$$\begin{aligned} \tau_j^{ab} = & 2 \exp \left( R(x) + T_j^+ + i\frac{\delta}{2} \right) \left\{ -\cosh R(x) \sin T_j^+ - e^{\frac{-\zeta}{2}} \sinh \frac{\eta_-}{2} \sin(\sigma vt - T_j^-) \right. \\ & \left. + i \left( \sinh R(x) \cos T_j^+ - e^{\frac{-\zeta}{2}} \cosh \frac{\eta_-}{2} \cos(\sigma vt - T_j^-) \right) \right\}. \end{aligned} \quad (3.7)$$

### 3.2.1 Properties of the Interaction Coefficient

The interaction coefficient  $A$  has properties similar to the properties of the  $S$ -matrix of the fundamental Toda particles. For type A and B breathers, the interaction coefficient (1.45) is given by

$$A_{(ab)} = -\frac{(\sigma_a - \sigma_b)^2 - (\sigma_a i v_a - \sigma_b i v_b)^2 - 4m^2 \sin^2(\frac{\pi}{h}(a-b))}{(\sigma_a + \sigma_b)^2 - (\sigma_a i v_a + \sigma_b i v_b)^2 - 4m^2 \sin^2(\frac{\pi}{h}(a+b))} \quad (3.8)$$

$$= \frac{\sin\left(\frac{\bar{\Theta}}{2} + \frac{\pi(a-b)}{2h}\right) \sin\left(\frac{\bar{\Theta}}{2} - \frac{\pi(a-b)}{2h}\right)}{\sin\left(\frac{\bar{\Theta}}{2} + \frac{\pi(a+b)}{2h}\right) \sin\left(\frac{\bar{\Theta}}{2} - \frac{\pi(a+b)}{2h}\right)}$$

where  $\bar{\Theta} = -i\Theta$ . For the type A breathers the interaction coefficient is,

$$A_{aa} = \frac{v^2}{(1+v^2) \cos^2(\frac{\theta_a}{2}) - 1} = \frac{1}{\cos^2 \frac{\theta_a}{2}} \frac{\frac{v_c^{(A)^2}}{v^2}}{\frac{v_c^{(A)^2}}{v^2} - 1}, \quad (3.9)$$

where the critical velocity  $v_c^{(A)}$ , when the interaction coefficient changes sign, is

$$v_c^{(A)} = \tan\left(\frac{\theta_a}{2}\right). \quad (3.10)$$

The term  $\theta_a$  is given by formula (1.42). In terms of rapidity difference  $\bar{\Theta}$  the coefficient can be written as,

$$A_{aa} = \frac{\sin^2(\frac{\bar{\Theta}}{2})}{\sin(\frac{\bar{\Theta}}{2} + \frac{\theta_a}{2}) \sin(\frac{\bar{\Theta}}{2} - \frac{\theta_a}{2})}.$$

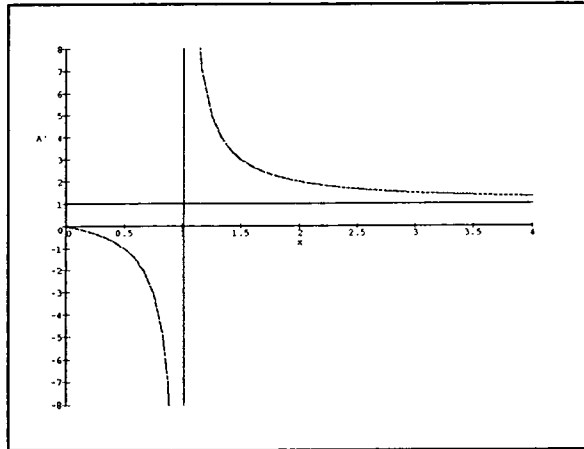


Fig. 3.1: The interaction  $A'$  is given by  $A' = A \cos^2(\theta/2)$ , the velocity as  $x = v^2/v_c^2$ .

It can have either positive or negative value. For breathers with constituent solitons of species  $a = \frac{h}{2}$ , the interaction coefficient never changes sign.

And for type B breathers,

$$A_{a\bar{a}} = (1 + v^2) \cos^2\left(\frac{\theta_a}{2}\right) - v^2 = \cos^2 \frac{\theta_a}{2} \left(1 - \frac{v^2}{v_c^{(B)^2}}\right), \quad (3.11)$$

where  $v_c^{(B)}$  is the critical velocity; in terms of rapidity difference  $\bar{\Theta}$ ,

$$A_{a\bar{a}} = \frac{\cos(\frac{\bar{\Theta}}{2} + \frac{\theta_a}{2}) \cos(\frac{\bar{\Theta}}{2} - \frac{\theta_a}{2})}{\cos^2(\frac{\bar{\Theta}}{2})}.$$

The critical velocity at which the interaction coefficient changes sign is, for a type B breather,

$$v_c^{(B)} = \frac{1}{v_c^{(A)}}. \quad (3.12)$$

As the case of  $S$ -matrices of fundamental Toda particles, these interaction coefficients admit a pole. For type A breathers,

$$v = v_c^{(A)} \quad \text{or,} \quad \bar{\Theta} = \frac{2\pi a}{h}$$

and, for type B breathers,

$$v \rightarrow \infty \quad \text{or,} \quad \bar{\Theta} = \pi.$$

It is readily seen that the pole of  $A_{aa}$  is exactly the fusing angle related to the process  $a + a \rightarrow \overline{(h - 2a)}$  of the fundamental particles [22]. Hollowood noted that the same fusing rule also applies to soliton fusings in  $a_n^{(1)}$  theories [35]. In fact, the fusing rule of fundamental particles applies also to all *simply laced* affine Toda solitons [45,102]. The fusing of two solitons of species  $a$  into  $\overline{(h - 2a)}$  hinted that the topological charge of type A breathers has to be found in the same representation as the topological charges of  $\overline{(h - 2a)}$  single solitons.

It is not surprising that, at the pole of the interaction coefficient, the breathers fail to exist. Using the breather  $\tau$ -function (3.5) with the interaction coefficient approaching its pole, the interaction term dominates. Hence, the solution falls into one of the vacuum solutions of the complex affine Toda potential,

$$\phi = -\frac{1}{\beta} \sum_{j=1}^n j \alpha_j \theta_+.$$

On the other hand, if the positions of the constituent solitons are simultaneously shifted by  $-\frac{\zeta}{2}$ , i.e. changing  $\eta \rightarrow \eta - \frac{\zeta}{2}$ , the breather turns into a static solution as the interaction coefficient approaches its pole.

Furthermore, the interaction coefficient  $A$  has the following general properties some of which are similar but not the same as the properties of the  $S$ -matrix,

- *Crossing symmetry*

$$A_{aa}(v) = A_{a\bar{a}}^{-1}\left(\frac{1}{v}\right) \quad \text{or,} \quad A_{aa}(\bar{\Theta}) = A_{a\bar{a}}^{-1}(\bar{\Theta} - \pi).$$

- *Evenness*

$$A(v) = A(-v) \quad \text{or,} \quad A(\bar{\Theta}) = A(-\bar{\Theta}).$$

- *Symmetry*

$$A_{a\bar{a}}(v) = A_{\bar{a}a}(v) \quad \text{or,} \quad A_{a\bar{a}}(\bar{\Theta}) = A_{\bar{a}a}(\bar{\Theta}).$$

- *Periodicity*

$$A(\bar{\Theta}) = A(\bar{\Theta} + 2\pi).$$

When the interaction coefficient is zero the  $\tau$ -functions do not give a well defined solution, as the  $\tau$ -functions will vanish at a particular point in space-time. For the type A breather a vanishing interaction coefficient can only occur for a vanishing rapidity difference. Thus the solution will be static and not “breath” at all. For the type B breather the situation less simple. It is best to look at the  $\tau$  function written in terms of the rapidity. The interaction coefficient vanishes for  $\bar{\Theta} = \pi \pm \theta_a$ . In that case the  $\tau$  function is

$$\begin{aligned} \tau_j^{a\bar{a}} = 1 + \exp & \left( 2mx \sin \frac{\theta_a}{2} \cos \frac{\bar{\Theta}}{2} - 2imt \sin \frac{\theta_a}{2} \sin \frac{\bar{\Theta}}{2} + \eta_a + i(\xi_a + j\theta_a) \right) \\ & + \exp \left( 2mx \sin \frac{\theta_{\bar{a}}}{2} \cos \frac{\bar{\Theta}}{2} + 2imt \sin \frac{\theta_{\bar{a}}}{2} \sin \frac{\bar{\Theta}}{2} + \eta_{\bar{a}} + i(\xi_{\bar{a}} + j\theta_{\bar{a}}) \right). \end{aligned}$$

To check whether this  $\tau$  function vanishes for any  $(x_0, t_0)$  given any parameter one has to write it separated into real and imaginary part again. For this introduce  $\Gamma(x) = -2mx \sin^2 \frac{\theta_a}{2} + \eta/2$ ,  $\Delta_j(t) = -2mt \sin \frac{\theta}{2} \cos \frac{\theta}{2} - \frac{1}{2}(\xi_- + j\theta_-)$  and finally

$T_j = \frac{1}{2}(\xi_+ + j\theta_+)$ . Then the  $\tau$  can be written as

$$\begin{aligned} \tau_j^{a\bar{a}} = 1 + 2e^{\Gamma(x)} & \left( \cos T_j \cosh \frac{\eta_-}{2} \cos \Delta_j(t) + \sin T_j \sinh \frac{\eta_-}{2} \sin \Delta_j(t) \right. \\ & \left. + i \left( \sin T_j \cosh \frac{\eta_-}{2} \cos \Delta_j(t) - \cos T_j \sinh \frac{\eta_-}{2} \sin \Delta_j(t) \right) \right). \end{aligned}$$

For  $\cos T_j \neq 0$  and  $\sin T_j \neq 0$  the imaginary part vanishes at

$$\tan \Delta_j(t_0) = \tan T_j \coth \frac{\eta_-}{2}.$$

So, the real part will vanish simultaneously at

$$e^{\Gamma(x_0)} = -\frac{1}{\sin \Delta_j(t_0)} \left[ \cos T_j \cot T_j \cosh \frac{\eta_-}{2} \coth \frac{\eta_-}{2} + \sin T_j \sinh \frac{\eta_-}{2} \right]^{-1}.$$

Because the time  $t_0$  is given via a  $\tan$  the sign of the left hand side of the equation defining  $x_0$  can always be chosen to be positive. If  $\cos T_j = 0$  or  $\sin T_j = 0$  one can find a point for which the  $\tau$  function vanishes similarly. The case  $\eta_- = 0$  is different in that one has to examine the ratio  $\frac{\tau_j^{a\bar{a}}}{\tau_0^{a\bar{a}}}$  instead. Again one finds a point where the  $\tau$  function vanishes.

### 3.2.1 Type A Breathers

The breathers with constituent solitons of the same species will have a negative interaction coefficient,  $A < 0$ , when  $v^2 < v_c^{(A)2}$  (see (3.1)). As mentioned before to have a well-defined  $\tau$  function for all  $(x, t)$  there are certain restrictions on the parameters  $\xi_+$  and  $\eta_-$ . These can be deduced from the  $\tau$  functions (3.6) and (3.7).

First the negative interaction coefficient will be discussed. Because one is actually interested in the ratio  $\frac{\tau_j^{a\bar{a}}}{\tau_0^{a\bar{a}}}$  the factor in front of the curly bracket in (3.7) is a constant phase. If the  $\tau$  function vanishes for a certain point  $(x_0, t_0)$  in space time, real and imaginary part have to vanish simultaneously. This implies that

$$\cosh R(x_0) = -e^{-\zeta/2} \frac{\sinh \frac{\eta_-}{2} \sin(\sigma v t_0 - T_j^-)}{\sin T_j^+} = -c_1 \sin \Delta_0 \quad (3.13)$$

and

$$\sinh R(x_0) = e^{-\zeta/2} \frac{\cosh \frac{\eta_-}{2} \cos(\sigma v t_0 - T_j^-)}{\cos T_j^+} = c_2 \cos \Delta_0 \quad (3.14)$$

with  $c_1 = e^{-\zeta/2 \frac{\sinh \frac{\eta_-}{2}}{\sin T_j^+}}$ ,  $c_2 = e^{-\zeta/2 \frac{\cosh \frac{\eta_-}{2}}{\cos T_j^+}}$  and  $\Delta_0 = \sin(\sigma v t_0 - T_j^-)$ . If  $\sin T_j^+ = 0$  the singularity can be avoided by choosing  $\eta_- = 0$ . Squaring and subtracting conditions (3.13) and (3.14) implies

$$c_1^2 \geq 1 \text{ and } c_2^2 \geq -1.$$

This means the singularity can be avoided if

$$c_1^2 < 1 \Rightarrow \sinh^2 \frac{\eta_-}{2} < |A| \sin^2 T_j^+. \quad (3.15)$$

This can only be achieved if

$$\xi_+ \neq (2\pi - j\theta_+) \bmod 2\pi, \quad j = 0, \dots, n. \quad (3.16)$$

This divides the range of  $\xi_+$  into several regions which will determine the topological charge of the breather. Furthermore, the maximum “distance” of separation  $\eta_-$  between the two constituent solitons is restricted by (3.15). For a non-singular solution one has to choose the minimum value of all  $\eta_-$ ’s allowed

$$-\min_j(\eta_c^j) < \eta_- < \min_j(\eta_c^j), \quad (3.17)$$

where,

$$\eta_c^j = \cosh^{-1} [2|A| \sin^2 T_j^+ + 1], \quad (3.18)$$

with  $j = 0, \dots, n$ .

For breathers of type A with positive interaction coefficient it can be derived in a similar way to that the parameters  $\xi_+$  must not have the following values,

$$\xi_+ = (\pi - j\theta_+) \bmod 2\pi, \quad j = 0, \dots, n. \quad (3.19)$$

The separation “distance”  $\eta_-$  is restricted as above with,

$$\eta_c^j = \operatorname{arcosh} [2A \cos^2 T_j^+ - 1] \Rightarrow A \cos^2 T_j^+ \geq 1.$$

As  $\eta_c^j$  are defined through an arcosh-function, these  $\eta_c^j$  in turn will restrict the allowed velocity of the constituent solitons because  $A$  depends on the velocity (3.9). Generally, not all  $v^2 > v_c^{(A)2}$  are allowed. In fact all velocities with absolute value

greater than the absolute value of the critical velocity are allowed if, for all  $j$ , the following is true,

$$\left[1 - \frac{\cos^2 T_j^+}{\cos^2(\frac{\theta_+}{4})}\right] < 0.$$

Otherwise the velocity is bounded from above by

$$v_c^{(A)2} < v^2 \leq \frac{v_c^{(A)2}}{\max_j \left[1 - \frac{\cos^2 T_j^+}{\cos^2(\frac{\theta_+}{4})}\right]}.$$

### 3.2.2 Type B Breathers

The constituent solitons of type B breathers are of opposite species. The negative interaction coefficient regime is accomplished with  $v^2 > v_c^{(B)2}$ . In this case, the real and imaginary part of the  $\tau$ -functions will be trivially zero for

$$\xi_+ = 0 \bmod 2\pi, \quad (3.20)$$

i.e. the parameter  $\xi_+$  must not be an integer multiple of  $2\pi$ . For each  $\tau_j$  to avoid zero, the separation parameter  $\eta_-$  is limited to take values between

$$-\min_j(\eta_c^j) < \eta_- < \min_j(\eta_c^j),$$

where,

$$\eta_c^j = \text{arcosh} \left[ 2|A| \sin^2 T_j^+ \right],$$

with  $j = 0, \dots, n$ . As for type A breathers, each  $\tau_j$  has its own  $\eta_c^j$ , and the smallest of these is taken as the limit.

For negative interaction coefficient the condition on  $\xi_+$  is

$$\xi_+ \neq \pi \bmod 2\pi.$$

But the restriction (3.18) implies with (3.11) that

$$v^2 \leq v_c^{(B)2} \left[ 1 - \frac{1}{\cos^2 \frac{\theta_+}{2} \cos^2 T_j^+} \right].$$



This can never be satisfied because  $\cos^2 \frac{\theta_+}{2} \cos^2 T_j^+ > 0$ . Contrary to the type A breathers, it is not possible to have type B breathers with positive interaction coefficient, or to have velocity  $v^2 < v_c^{(B)2}$ . Since the  $\tau$ -functions would necessarily pass the origin, this would lead to a solution which is not well-defined.

### 3.3 The Topological Charges

Unfortunately, the calculation of topological charges of multi-soliton solutions used in [37] is not applicable for breathers because the two solitons constituting the breather only separate a finite “distance” from each other.

Still, for the type A breather it is not too difficult to deduce the number of distinct topological charges in the fashion of [37]. The number of the topological charges is determined by the number of sectors of allowed values of  $\xi_+$ . From the expression for forbidden  $\xi_+$ 's, (3.16) or (3.19), it follows that one looks for the smallest number  $p$  for which

$$\frac{2ap}{h} = k \text{ with } p, k \in \mathbb{N}. \quad (3.21)$$

Here  $a$  is the species of the constituent solitons and  $h$  is the Coxeter number.

With  $2\tilde{a} = \frac{2a}{\gcd(2a, h)}$ ,  $\tilde{h} = \frac{h}{\gcd(2a, h)}$  then (3.21) can be rewritten as

$$2\tilde{a}p = \tilde{h}k.$$

Because  $2\tilde{a}$  and  $\tilde{h}$  are coprime it follows that  $p = \tilde{h}$  and  $k = 2\tilde{a}$ . Thus the range of the allowed values for  $\xi_+$  is divided into  $\tilde{h}$  sectors. This leads to the following formula for the maximum number of topological charges of type A breather with constituent solitons of species  $a$

$$\tilde{h} = \frac{h}{\gcd(2a, h)}. \quad (3.22)$$

This argument holds independently of the sign of the interaction coefficient. The argument of  $f_j(x, t)$  can only change when  $f_j(x, t)$  is undefined or zero, hence the topological charge within each sector is constant. It turns out that in each of these sectors, the topological charges take a different unique value. These topological charges are related by permutation of the roots  $\alpha_j$  for  $j = 0, \dots, n$ . The topological

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The remaining sections of this chapter are identical with the publication [95].

charge of a specific sector will be determined first, and is called the *highest charge* [37]. Then it will be shown that in all other sectors, the topological charges will be different. This means that  $\tilde{h}$  is indeed the number of topological charges associated with a breather.

In the following, calculation of the highest charge will be performed for a type A breather with negative interaction coefficient. The calculation for positive interaction coefficient is the same and will not be presented here. To calculate the highest topological charge, one has to employ a little trick first to simplify the breather  $\tau$ -function. The type A breather  $\tau$ -function is given by

$$\tau_j^{(aa)} = 1 + \exp[\sigma_a(x + ivt) + \rho + ij\theta_a] + \exp[\sigma_a(x - ivt) + \rho' + ij\theta_a] - \exp[\zeta + 2\sigma_a x + \rho_+ + 2ij\theta_a].$$

First choose  $t = 0$  because the topological charge does not depend on the time. Second choose  $\rho = -\zeta/2 + \hat{\rho}$  and  $\rho' = -\zeta/2 + \hat{\rho}'$ , this corresponds to a simultaneous shift of the constituent solitons to the left. By this shifting, the last term in the breather  $\tau$ -function will not depend on the interaction coefficient  $A$ ,

$$\tau_j^{(aa)} = 1 + \exp(\sigma_a x + ij\theta_a - \zeta/2)(e^{\hat{\rho}} + e^{\hat{\rho}'}) - \exp(2\sigma_a x + \hat{\rho}_+ + 2ij\theta_a).$$

With  $\mu_a(2j) = \frac{4\pi a j}{h} \bmod 2\pi$ , the limits  $|x| \rightarrow \infty$  of  $f_j$  will give

$$\begin{aligned} \lim_{x \rightarrow \infty} f_j &= e^{i\mu_a(2j)}, \\ \lim_{x \rightarrow -\infty} f_j &= 1. \end{aligned} \tag{3.23}$$

Moreover one can take the limit  $\zeta$  approaching  $+\infty$ , this corresponds to choosing the velocity  $v$  very near to  $v_c^{(A)}$ . As long as  $v$  is not equal to  $v_c^{(A)}$  the breather solution is well-defined by construction. Write  $y = e^{2\sigma_a x}$ , then provided one does not take the limit  $x \rightarrow \infty$ , the  $y^{1/2}$  term can be dropped,

$$\begin{aligned} \tau_j^{(aa)} &= 1 + y^{1/2} \exp(ij\theta_a - \zeta/2)(e^{\hat{\rho}} + e^{\hat{\rho}'}) - y \exp(\hat{\rho}_+ + 2ij\theta_a) \\ &= 1 - y \exp(\hat{\rho}_+ + i\mu_a(2j)). \end{aligned} \tag{3.24}$$

By splitting the ratio  $f_j$  into its real and imaginary part one can now easily show that  $f_j$  traces out a clockwise curve in the complex plane, i.e. the winding number  $k$

is zero. To see this take  $\hat{\rho} = \hat{\rho}' = i(\pi - \frac{\varepsilon}{2})$  where  $\varepsilon$  is a real, positive and infinitesimal parameter,

$$\tau_j = 1 - y \exp(i(\mu_a(2j) - \varepsilon)).$$

Then the ratio  $f_j$  can be written as

$$f_j = \frac{1}{|1 - ye^{\hat{\rho}+}|^2} [1 - y(\cos(\mu_a(2j) - \varepsilon) + \cos(\varepsilon)) + y^2 \cos(\mu_a(2j)) \\ + i\{-y(\sin(\mu_a(2j) - \varepsilon) + \sin(\varepsilon)) + y^2 \sin(\mu_a(2j))\}].$$

The only zeros for the imaginary part occur when  $y = 0$  and

$$y = \frac{\sin(\mu_a(2j) - \varepsilon) + \sin(\varepsilon)}{\sin(\mu_a(2j))},$$

with  $\mu_a(2j) \neq 0$  or  $\pi$ . For small  $\varepsilon$  this is

$$y = 1 + \varepsilon \frac{1 - \cos(\mu_a(2j))}{\sin(\mu_a(2j))} + O(\varepsilon^2). \quad (3.25)$$

Now inserting (3.25) into the real part of  $f_j$  results in,

$$\Re(f_j |1 - ye^{\hat{\rho}+}|^2) = -2\varepsilon \frac{1 - \cos(\mu_a(2j))}{\sin(\mu_a(2j))} + O(\varepsilon^2).$$

One should also observe that in the small  $\varepsilon$  regime the imaginary part behaves for small  $y$  like

$$\Im(f_j |1 - ye^{\hat{\rho}+}|^2) = -y \sin(\mu_a(2j)) + \dots$$

So, for  $0 < \mu_a(2j) < \pi$ , the curve starts at  $(1, 0)$  with a negative imaginary part and crosses the negative part of the real line. For  $\pi < \mu_a(2j) < 2\pi$ , it starts at  $(1, 0)$  with a positive imaginary part and crosses the positive part of the real line. In any case, it winds around the origin in the clockwise sense. Thus, the change of argument of  $f_j$  is given by  $\mu_a(2j) - 2\pi$ . The explicit formula for the *highest topological charge* is therefore determined by (1.51)

$$q_a^{(1)} = \sum_{j=0}^n \frac{2a(h-j) \bmod h}{h} \alpha_j. \quad (3.26)$$

In the summation above the extended root  $\alpha_0$  is included for convenience in the permutation of the simple roots and  $\alpha_0$ . As mentioned previously, this result does not depend on the sign of the interaction coefficient.

From this highest charge, one can obtain all the other charges as follows. Suppose initially the value of  $\xi_+$  is chosen. Then, making a shift of  $\frac{4\pi a}{h}$  on this  $\xi_+$  amounts to sending the breather solution to a different sector of  $\xi_+$ . Successive applications of this shift will bring the breather solution to every allowed sector of  $\xi_+$ . With the  $\tilde{h}^{th}$  application it will return to the original sector. Recall that with (3.24) the breather solution is given by,

$$\phi = \frac{i}{\beta} \sum_{j=0}^n \alpha_j \ln(1 - \omega_a^{2j} y e^{\hat{\rho}_+}).$$

Making the shift  $\xi_+ \rightarrow \xi_+ - \frac{4\pi a}{h}$  in the above solution gives,

$$\phi = \frac{i}{\beta} \sum_{j=0}^n \alpha_j \ln(1 - \omega_a^{2j} y e^{\hat{\rho}_+ - \frac{4i\pi a}{h}}) = \frac{i}{\beta} \sum_{j=0}^n \alpha_{j+1} \ln(1 - \omega_a^{2j} y e^{\hat{\rho}_+}).$$

Thus this shifting is the same as cyclically permuting the roots  $\alpha_j$  for all  $j = 0, \dots, n$ . And hence, each shifting results in a different topological charge. Since the maximum number one can shift  $\xi_+$  is  $\tilde{h}$  times, then  $\tilde{h}$  is exactly the number of topological charges of the breather solution. The expression for all the topological charges is

$$q_a^{(k)} = \sum_{j=0}^n \frac{2a(h-j) \bmod h}{h} \alpha_{(j+k-1)}, \quad k = 1, 2, \dots, \tilde{h}, \quad (3.27)$$

where the roots  $\alpha_j$  are labelled modulo  $h$ .

This is analogous to the one soliton case [37]. Furthermore, as explained in the same paper, all these topological charges lie in the same representation because they are related by a Weyl transformation as will be shown in the next subsection.

For the type B breather it has been determined in a preceding calculation that there is only one sector of allowed values for  $\xi_+$ . The only possible way for the topological charge to change is when the ratio  $f_j$  is not well-defined, i.e  $\xi_+$  changes from one sector to another. So, in this case there cannot be a change in the topological charge. The only open question now is what value the topological charge takes. To determine this one simply follows the previous prescription [37]. The  $\tau$ -functions for the type B breather are given by

$$\begin{aligned} \tau_j^{(a\bar{a})} = & 1 + \exp[\sigma_a(x + ivt) + \rho + ij\theta_a] + \exp[\sigma_a(x - ivt) + \rho' - ij\theta_a] \\ & - \exp[\zeta + 2\sigma_a x + \rho_+], \end{aligned}$$

where  $\bar{a} = h - a$ . Because the topological charge is time independent one can set  $t = 0$ . Also, one can substitute  $\exp(\sigma_a x) = z$ ,  $\rho = \rho' = i(\pi + \frac{\epsilon}{2})$  with  $\epsilon \in \mathbb{R}$  and infinitesimal. The  $\tau$ -function is then given in the compact form

$$\tau_j^{(a\bar{a})} = 1 - 2z \cos(j\theta_a) e^{i\frac{\epsilon}{2}} - z^2 e^{\zeta + i\epsilon}.$$

Let  $f_j$  be defined as before. The start and end point of the curve traced out by  $f_j$  as  $x$  goes from  $-\infty$  to  $\infty$  are, in this case the same,  $f_j(x = \pm\infty) = 1$ . Solving an equation for the imaginary part of  $f_j$  one finds that these are also the only points for which the imaginary part vanishes. Therefore the winding number  $k$  is zero, because the curve cannot wrap around the origin. Moreover, since the change of arguments of  $f_j$  as  $x$  goes from  $-\infty$  to  $\infty$  is zero, the topological charge of any type B breather is deduced to be zero. In a sense, type B breathers are *sine-Gordon like* breathers. The constituent solitons are of opposite topological charges such that the resulting breather has zero topological charge. In fact, as will be discussed in the next section, type B breathers do not come from a sine-Gordon embedding in the theory.

### 3.3.1 Topological Charge and Representation Space

It is natural to expect that the topological charges which have been derived in the previous calculation lie in the tensor product representation of the fundamental representation associated with the topological charges of the constituent solitons. In fact, for the type A breather, with the exception for breathers built from species  $(n+1)$  in the  $a_{2n+1}^{(1)}$  cases, the topological charges lie in the fundamental representation which is a component of the Clebsch-Gordan decomposition of the tensor product representation. For type B breathers and the exceptional cases above, the topological charge (which is zero) lies in the singlet representation component of the Clebsch-Gordan decomposition of the tensor product representation.

For the non-zero highest topological charge, the first step is to show that it lies in the  $\mathcal{R}_{\lambda_{2a \bmod h}}$  fundamental representation. This will be shown using a combination of Weyl transformations [37]. Then the second step is to show the other topological charges are related to the highest charge by a special Coxeter element of the Weyl group. It is convenient to write the highest charge (3.26) as

$$q_a^{(1)} = \sum_{j=0}^n \frac{bj \bmod h}{h} \alpha_j, \quad (3.28)$$

where  $b = h - (2a \bmod h)$ . Because of the  $Z_2$  symmetry of the simple roots, it is necessary only to consider the case  $b \leq \left\lfloor \frac{h}{2} \right\rfloor$ . The notation  $[x]$  means the largest integer less than or equal to  $x$ , hence for  $h$  even,  $\left\lfloor \frac{h}{2} \right\rfloor = \frac{h}{2}$ , and for  $h$  odd,  $\left\lfloor \frac{h}{2} \right\rfloor = \frac{h-1}{2}$ . Furthermore, (3.28) can be rewritten in terms of the fundamental weights  $\lambda_j$  defined by  $\frac{2\lambda_j \cdot \alpha_k}{\alpha_k^2} = \delta_{jk}$  as follows,

$$\begin{aligned} q_a^{(1)} = & \frac{1}{h} \{2[b \bmod h] - [2b \bmod h]\} \lambda_1 + \frac{1}{h} \{2[bn \bmod h] - [b(n-1) \bmod h]\} \lambda_n \\ & + \sum_{j=1}^{n-1} \frac{1}{h} \{2[bj \bmod h] - [b(j-1) \bmod h] - [b(j+1) \bmod h]\} \lambda_j. \end{aligned} \quad (3.29)$$

Then the following can be demonstrated easily,

$$q_a^{(1)} \cdot \alpha_j = \begin{cases} 1 & j = n, \\ 0 \text{ or } -1 & j = n-1, \\ 0 \text{ or } -1 \text{ or } 1 & 1 \leq j < n-1. \end{cases}$$

The part  $q_a^{(1)} \cdot \alpha_j = -1$  for  $j < n-1$  will be demonstrated in the following. Let,  $bj = ch + d$  where  $d < b$  and  $c \geq 0$ , thus  $j = 1$  is excluded. Then with (3.29) one finds that,

$$\begin{aligned} q_a^{(1)} \cdot \alpha_j &= \frac{1}{h} \{2[bj \bmod h] - [b(j-1) \bmod h] - [b(j+1) \bmod h]\} \\ &= \frac{1}{h} \{2d - (d - b + h) - (d + b)\} = -1. \end{aligned}$$

There are  $(b-1)$  terms of  $q_a^{(1)} \cdot \alpha_j = -1$  for  $j < n-1$  since this happens only when  $d < b$ . Furthermore, it is straightforward to see that for  $1 < j < n-1$

$$q_a^{(1)} \cdot \alpha_j = -1 \implies q_a^{(1)} \cdot \alpha_{j-1} = 1.$$

Thus, if the scalar products of  $q_a^{(1)}$  with the simple roots  $\{\alpha_j\}$  are written as a row vector, it has the entry 1 at the  $n^{th}$  position and there are  $(b-1)$  pairs of  $(1, -1)$  to the left of it,

$$q_a^{(1)} \cdot \{\alpha_j\} = (0, \dots, 0, 1, -1, 0, \dots, 1, -1, 1, -1, \dots, 0, 1),$$

the  $j^{th}$  entry of the row vector on the right-hand side is  $q_a^{(1)} \cdot \alpha_j$ .

It is also elementary to see the following. Suppose a weight  $\gamma_1$  has a scalar product with the simple roots  $\gamma_1 \cdot \{\alpha_j\} = (0, \dots, 0, 1, -1, 0, \dots, 0)$ . Consider the Weyl reflection  $r$  with respect to the simple root  $\alpha_k$  where  $\gamma_1 \cdot \alpha_k = -1$ . The action of  $r$  on  $\gamma_1$  will shift the pair  $(1, -1)$  in  $\gamma_1 \cdot \{\alpha_j\}$  one step to the right, i.e.  $r : \gamma_1 \longrightarrow \gamma'_1$  with

$$\gamma'_1 \cdot \{\alpha_j\} = (0, \dots, 0, 0, 1, -1, \dots, 0).$$

For a weight  $\gamma_2$  which has  $\gamma_2 \cdot \{\alpha_j\} = (0, \dots, 0, 1, -1, 1, \dots, 0)$ , consider the Weyl reflection  $r'$  with respect to the simple root  $\alpha_k$  where  $\gamma_2 \cdot \alpha_k = -1$ . The action of  $r'$  on  $\gamma_2$  will give  $\gamma'_2$  where,

$$\gamma'_2 \cdot \{\alpha_j\} = (0, \dots, 0, 0, 1, 0, \dots, 0).$$

So, using a combination of these Weyl transformations,  $q_a^{(1)}$  can be transformed into a fundamental weight,  $q_a^{(1)} \longrightarrow \lambda$ , where

$$\lambda \cdot \{\alpha_j\} = (0, \dots, 0, 1, 0, \dots, 0).$$

Since there are  $(b-1)$  pairs of  $(1, -1)$  in the row vector  $q_a^{(1)} \cdot \{\alpha_j\}$ , then after these combinations of Weyl transformations the entry 1 will appear at the position  $n - (b-1) = 2a \bmod h$ . Hence the highest topological charge  $q_a^{(1)}$  lies in the fundamental representation  $\mathcal{R}_{\lambda_{2a \bmod h}}$ .

Recall that the rest of the topological charges are obtained by cyclically permuting the simple roots and  $\alpha_0$ , (3.27). This cyclic permutation is the same as the action of the following Coxeter element of the Weyl group on  $q_a^{(1)}$ ,

$$\omega_{tc} = r_1 r_2 \dots r_n, \quad (3.30)$$

where  $r_j$  is a Weyl reflection with respect to the simple root  $\alpha_j$ . Then, the topological charges are related to the highest charge by,

$$q_a^{(k)} = \omega_{tc}^{k-1}(q_a^{(1)}). \quad (3.31)$$

Note that the ordering of Weyl reflections above is special, other orderings do not necessarily relate one topological charge to another. The relation (3.31) is straightforward to see using the fact that,

$$\omega_{tc}(\alpha_j) = \alpha_{j+1} \quad \text{for } j = 0, 1, \dots, n \quad (3.32)$$

where the simple roots and  $\alpha_0$  are labelled modulo  $h$ . Further examination of (3.27) shows that the set of topological charges  $\{q_a^{(k)}\}$  coincides with the topological charges of the species  $2a \bmod h$  single solitons.

The next task is to show that  $\mathcal{R}_{\lambda_{2a \bmod h}}$  is a component of the Clebsch-Gordan decomposition of  $\mathcal{R}_{\lambda_a} \otimes \mathcal{R}_{\lambda_a}$ . This will be shown using a conjecture attributed to Parthasarathy, Ranga Rao and Varadarajan [103]. The PRV conjecture may be stated as follows: let  $\bar{\gamma}$  be a unique dominant weight of the Weyl orbit of  $\gamma = \lambda + \omega\mu$  for any  $\omega$  in the Weyl group and  $\lambda, \mu$  are highest weights, then  $\mathcal{R}_{\bar{\gamma}}$  appears with multiplicity of at least one in the decomposition of  $\mathcal{R}_{\lambda} \otimes \mathcal{R}_{\mu}$ , where  $\mathcal{R}_{\lambda}$  and  $\mathcal{R}_{\mu}$  are finite dimensional irreducible representations with highest weights  $\lambda$  and  $\mu$  respectively. This conjecture has been proved recently [104]; it was first used in the context of affine Toda theories by Braden [105].

For convenience of calculation, one can write the fundamental weights of the Lie algebra  $A_n$  as follows,

$$\lambda_a = \sum_{j=0}^a \frac{(h-a)j}{h} \alpha_j + \sum_{j=a+1}^n \frac{a(h-j)}{h} \alpha_j. \quad (3.33)$$

By the  $Z_2$  symmetry of the simple roots of  $A_n$ , one has to consider only the case  $a \leq [\frac{h}{2}]$ . Choose  $\omega$  to be the Coxeter element defined in (3.30). Then, remembering the action of this Coxeter element on the simple roots, c.f. (3.32), it is easy to show that

$$\lambda_a + \omega_{tc}^a \lambda_a = \lambda_{2a}.$$

It is obvious that  $\lambda_{2a}$  is a unique dominant weight of the Weyl orbit. Thus by the PRV conjecture  $\mathcal{R}_{\lambda_{2a \bmod h}} \subset \mathcal{R}_{\lambda_a} \otimes \mathcal{R}_{\lambda_a}$ .

This completes the claim that all the topological charges lie in the same fundamental representation  $\mathcal{R}_{\lambda_{2a \bmod h}}$  which is an irreducible component of  $\mathcal{R}_{\lambda_a} \otimes \mathcal{R}_{\lambda_a}$ ,

$$\{q_a^{(k)}\} \in \mathcal{R}_{\lambda_{2a \bmod h}} \subset \mathcal{R}_{\lambda_a} \otimes \mathcal{R}_{\lambda_a}.$$

However, as noted in the previous calculation, the number of topological charges is  $\tilde{h} = \frac{h}{\gcd(2a, h)}$  which is generally less than the dimension of  $\mathcal{R}_{\lambda_{2a \bmod h}}$ . So, the topological charges of type A breathers, normally do not fill the fundamental representation  $\mathcal{R}_{\lambda_{2a \bmod h}}$ . Only particular combinations of the topological charge of the constituent



solitons can make up a breather. A special case of the type A breather is when the constituent solitons come from the fundamental representation  $\mathcal{R}_{\lambda_a}$  which is self-conjugate, this happens for  $\mathcal{R}_{\lambda_{n+1}}$  in the representation of  $A_{2n+1}$ . This breather belongs to both type A and B.

For the type B breathers and the exceptional case above, the fundamental representations of its constituent solitons are conjugates of each other (or self-conjugate). Thus, the topological charge of these breathers will lie in the tensor product  $\mathcal{R}_{\lambda_a} \otimes \mathcal{R}_{\lambda_{h-a}}$ . Using the PRV conjecture as before, it can be shown that

$$\lambda_a + \omega_{tc}^a \lambda_{h-a} = 0.$$

Hence, the trivial *singlet* representation appears in the Clebsch-Gordan decomposition of this tensor product. It is in this singlet representation that the topological charge lies.

### 3.4 Sine-Gordon Embedding

Automorphisms of the Dynkin diagram can be used to reduce an affine Toda theory to another affine Toda theory with fewer scalar fields [17]. Using this reduction method, Sasaki noted that in the  $a_n^{(1)}$  affine Toda theories with a real coupling parameter, there are ways to reduce some members of the  $a_n^{(1)}$  family to the  $a_1^{(1)}$  theory, i.e. the sinh-Gordon theory [106]. The same procedure can be applied in the case of complex Toda theories. Define the solution to the equation of motion (1.11) as

$$\phi = \mu\psi, \tag{3.34}$$

where  $\mu$  is some vector to be determined. Then (1.11) becomes

$$\mu \partial^2(\beta\psi) = im^2 \sum_{j=1}^n \alpha_j \left( e^{i\beta\alpha_j \cdot \mu\psi} - e^{i\beta\alpha_0 \cdot \mu\psi} \right). \tag{3.35}$$

The aim is to reduce (3.35) above into the sine-Gordon equation of motion by choosing a suitable  $\mu$

$$\mu \partial^2(\beta\psi) = im^2 \mu \left( e^{i\beta\psi} - e^{-i\beta\psi} \right) = -2m^2 \mu \sin(\beta\psi). \tag{3.36}$$

There are two kinds of reductions. A *direct reduction* results when several nodes of the affine Dynkin diagram which do not have a direct link are identified. When linked nodes are transposed, this results in a *non-direct reduction*.

One can reduce the  $a_{2n+1}^{(1)}$  theories to the  $a_1^{(1)}$  theory using a direct reduction by choosing  $\mu$  as follows [106] ,

$$\mu_1 = \alpha_1 + \alpha_3 + \dots + \alpha_{2n-1} + \alpha_{2n+1}.$$

The vector  $\mu_1$  is an invariant vector under the  $Z_{n+1}$  symmetry which identifies  $\alpha_j \rightarrow \alpha_{j+2}$ . Projecting the simple roots of  $a_{2n+1}^{(1)}$  to  $\mu_1$  subspace gives the simple roots of  $a_1^{(1)}$  with multiplicity  $(n+1)$ ,

$$\alpha_j \cdot \mu_1 = 2 \text{ or } -2,$$

for  $j$  odd or even respectively. There are two choices of  $\mu$  for the non-direct reduction [106],

$$\mu_2 = \alpha_1 + \alpha_2 + \alpha_5 + \alpha_6 + \dots + \alpha_{4n-3} + \alpha_{4n-2},$$

$$\mu_3 = \alpha_2 + \alpha_3 + \alpha_6 + \alpha_7 + \dots + \alpha_{4n-2} + \alpha_{4n-1},$$

in the above,  $\mu_3$  is obtained from  $\mu_2$  by cyclically permuting the simple roots of  $a_{4n-1}^{(1)}$  once. Together with the vector  $\mu_1$ , these three vectors are invariant under the  $Z_n$  symmetry which identifies  $\alpha_j \rightarrow \alpha_{j+4}$ . The simple roots of  $a_{4n-1}^{(1)}$  can be projected to  $\mu_2$  or  $\mu_3$  giving the simple roots of  $a_1^{(1)}$  with multiplicity  $2n$ .

In terms of the single soliton  $\tau$ -functions (1.41) , direct reduction forces some  $\tau$ -functions to be equal leaving only two different  $\tau$ -functions,

$$\tau_0^{(a)} = \tau_2^{(a)} = \dots = \tau_{2n}^{(a)} \quad \text{and} \quad \tau_1^{(a)} = \tau_3^{(a)} = \dots = \tau_{2n-1}^{(a)}, \quad (3.37)$$

with

$$\tau_j^{(a)} = 1 + \omega_a^j e^{(\Omega_a + \rho)}.$$

Since  $\omega_a^j = \exp(\frac{2i\pi a}{h} j)$ , it is clear that for the  $a_{2n+1}^{(1)}$  theories, only solitons of species  $a = (n+1)$  are the *true* sine-Gordon solitons embedded in the theory. For the non-direct reductions, one has to have the following condition for the  $\tau$ -functions. Using

$\mu_2$  yields,

$$\tau_0^{(a)} = \tau_3^{(a)} = \tau_4^{(a)} = \dots = \tau_{4n-4}^{(a)} = \tau_{4n-1}^{(a)},$$

$$\tau_1^{(a)} = \tau_2^{(a)} = \tau_5^{(a)} = \dots = \tau_{4n-3}^{(a)} = \tau_{4n-2}^{(a)},$$

and for the choice of  $\mu_3$ ,

$$\tau_0^{(a)} = \tau_1^{(a)} = \tau_4^{(a)} = \dots = \tau_{4n-4}^{(a)} = \tau_{4n-3}^{(a)},$$

$$\tau_2^{(a)} = \tau_3^{(a)} = \tau_6^{(a)} = \dots = \tau_{4n-2}^{(a)} = \tau_{4n-1}^{(a)}.$$

These conditions on the  $\tau$ -functions of  $a_{4n-1}^{(1)}$  will never be satisfied. This is because for  $h = 4n$ , the factor  $\omega_a^j$  cannot be equal to  $\omega_a^{j+1}$  since  $j$  and  $(j+1)$  are coprime.

Thus, the solitons associated with the middle spot of the  $A_{2n+1}$  Dynkin diagram are the only sine-Gordon solitons embedded in the  $a_{2n+1}^{(1)}$  affine Toda theories. Hence, these solitons can bind together resulting in sine-Gordon breathers, i.e. type A breathers with zero topological charge. Note also that type B breathers by the above definitions are not formed from any sine-Gordon embedded solitons.

### 3.5 Examples

In this section the case of  $a_3^{(1)}$  and  $a_4^{(1)}$  breathers will be given as examples.

#### 3.5.1 $a_3^{(1)}$

The number above each spot on the Dynkin diagram are the number of topological charges of the type A breathers constructed from solitons

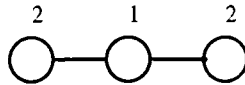


Fig. 3.2: Dynkin diagram of  $a_3$  with the number of breathers for each node

associated with each spot (see fig. (3.2)). The topological charges of the type A breather,  $q_a^{(k)}$ , are listed below. The subscript denotes the species of the constituent solitons and the superscript labels the topological charges,  $q_a^{(1)}$  is the *highest* topological charge.

$$q_1^{(1)} = \frac{1}{2}\alpha_1 + \frac{1}{2}\alpha_3,$$

$$q_1^{(2)} = -\frac{1}{2}\alpha_1 - \frac{1}{2}\alpha_3,$$

and,

$$q_2^{(1)} = 0.$$

The topological charges  $\{q_3\}$  are the same as  $\{q_1\}$  and are not listed above. All type B breathers have zero topological charge.

The topological charges  $\{q_1\}$ , and hence also  $\{q_3\}$ , lie in the second fundamental representation,  $\mathcal{R}_{\lambda_2} \subset \mathcal{R}_{\lambda_1} \otimes \mathcal{R}_{\lambda_1}$  or  $\mathcal{R}_{\lambda_3} \otimes \mathcal{R}_{\lambda_3}$ . The dimension of  $\mathcal{R}_{\lambda_2}$  is 6, and there are only 2 topological charges for  $\phi_{11}$  or  $\phi_{33}$  breathers. Thus, these topological charges do not fill up  $\mathcal{R}_{\lambda_2}$ . For  $q_2$ , as explained in the previous section, this is an embedded sine-Gordon breather, hence  $q_2 = 0$ .

Since topological charges are conserved quantities, it follows that for both type A breathers and type B breathers  $q_a$  is equal to the sum of the topological charges of the constituent solitons. Thus only a special combination of constituent solitons can make up a breather.

### 3.5.2 $a_4^{(1)}$

The solitons of this theory are associated with the nodes of the Dynkin diagram for  $A_4$ .

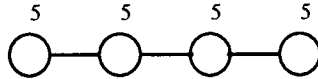


Fig. 3.3: Dynkin diagram of  $a_4$  with the number of breathers for each node

The number of breathers for each node can be found in figure (3.3). The topological charges of type A breathers in  $a_4^{(1)}$  are listed as follows. Breathers from species  $a = 1$  solitons:

$$\begin{aligned}
 q_1^{(1)} &= \frac{3}{5}\alpha_1 + \frac{1}{5}\alpha_2 + \frac{4}{5}\alpha_3 + \frac{2}{5}\alpha_4, \\
 q_1^{(2)} &= -\frac{2}{5}\alpha_1 + \frac{1}{5}\alpha_2 - \frac{1}{5}\alpha_3 + \frac{2}{5}\alpha_4, \\
 q_1^{(3)} &= -\frac{2}{5}\alpha_1 - \frac{4}{5}\alpha_2 - \frac{1}{5}\alpha_3 - \frac{3}{5}\alpha_4, \\
 q_1^{(4)} &= \frac{3}{5}\alpha_1 + \frac{1}{5}\alpha_2 - \frac{1}{5}\alpha_3 + \frac{2}{5}\alpha_4, \\
 q_1^{(5)} &= -\frac{2}{5}\alpha_1 + \frac{1}{5}\alpha_2 - \frac{1}{5}\alpha_3 - \frac{3}{5}\alpha_4.
 \end{aligned}$$

Breathers from species  $a = 2$  solitons:

$$\begin{aligned}
 q_2^{(1)} &= \frac{1}{5}\alpha_1 + \frac{2}{5}\alpha_2 + \frac{3}{5}\alpha_3 + \frac{4}{5}\alpha_4, \\
 q_2^{(2)} &= -\frac{4}{5}\alpha_1 - \frac{3}{5}\alpha_2 - \frac{2}{5}\alpha_3 - \frac{1}{5}\alpha_4, \\
 q_2^{(3)} &= \frac{1}{5}\alpha_1 - \frac{3}{5}\alpha_2 - \frac{2}{5}\alpha_3 - \frac{1}{5}\alpha_4, \\
 q_2^{(4)} &= \frac{1}{5}\alpha_1 + \frac{2}{5}\alpha_2 - \frac{2}{5}\alpha_3 - \frac{1}{5}\alpha_4, \\
 q_2^{(5)} &= \frac{1}{5}\alpha_1 + \frac{2}{5}\alpha_2 + \frac{3}{5}\alpha_3 - \frac{1}{5}\alpha_4.
 \end{aligned}$$

Breathers from species  $a = 3$  solitons:

$$\begin{aligned}
 q_3^{(1)} &= \frac{4}{5}\alpha_1 + \frac{3}{5}\alpha_2 + \frac{2}{5}\alpha_3 + \frac{1}{5}\alpha_4, \\
 q_3^{(2)} &= -\frac{1}{5}\alpha_1 + \frac{3}{5}\alpha_2 + \frac{2}{5}\alpha_3 + \frac{1}{5}\alpha_4, \\
 q_3^{(3)} &= -\frac{1}{5}\alpha_1 - \frac{2}{5}\alpha_2 + \frac{2}{5}\alpha_3 + \frac{1}{5}\alpha_4, \\
 q_3^{(4)} &= -\frac{1}{5}\alpha_1 - \frac{2}{5}\alpha_2 - \frac{3}{5}\alpha_3 + \frac{1}{5}\alpha_4, \\
 q_3^{(5)} &= -\frac{1}{5}\alpha_1 - \frac{2}{5}\alpha_2 - \frac{3}{5}\alpha_3 - \frac{4}{5}\alpha_4.
 \end{aligned}$$

Breathers from species  $a = 4$  solitons:

$$\begin{aligned}
q_4^{(1)} &= \frac{2}{5}\alpha_1 + \frac{4}{5}\alpha_2 + \frac{1}{5}\alpha_3 + \frac{3}{5}\alpha_4, \\
q_4^{(2)} &= -\frac{3}{5}\alpha_1 - \frac{1}{5}\alpha_2 + \frac{1}{5}\alpha_3 - \frac{2}{5}\alpha_4, \\
q_4^{(3)} &= \frac{2}{5}\alpha_1 - \frac{1}{5}\alpha_2 + \frac{1}{5}\alpha_3 + \frac{3}{5}\alpha_4, \\
q_4^{(4)} &= -\frac{3}{5}\alpha_1 - \frac{1}{5}\alpha_2 - \frac{4}{5}\alpha_3 - \frac{2}{5}\alpha_4, \\
q_4^{(5)} &= \frac{2}{5}\alpha_1 - \frac{1}{5}\alpha_2 + \frac{1}{5}\alpha_3 - \frac{2}{5}\alpha_4.
\end{aligned}$$

These topological charges lie in the following fundamental representation,

$$\begin{aligned}
\{q_1^{(k)}\} &\in \mathcal{R}_{\lambda_2} \subset \mathcal{R}_{\lambda_1} \otimes \mathcal{R}_{\lambda_1}, \\
\{q_2^{(k)}\} &\in \mathcal{R}_{\lambda_4} \subset \mathcal{R}_{\lambda_2} \otimes \mathcal{R}_{\lambda_2}, \\
\{q_3^{(k)}\} &\in \mathcal{R}_{\lambda_1} \subset \mathcal{R}_{\lambda_3} \otimes \mathcal{R}_{\lambda_3}, \\
\{q_4^{(k)}\} &\in \mathcal{R}_{\lambda_3} \subset \mathcal{R}_{\lambda_4} \otimes \mathcal{R}_{\lambda_4}.
\end{aligned}$$

There is no sine-Gordon embedding in this case and the topological charges of all type B breathers are zero.

### 3.6 Conclusions

Following the example of the sine-Gordon theory, classical oscillating soliton solutions of the  $a_n^{(1)}$  affine Toda theories have been constructed as bound states of soliton pairs. These breathers are classified by the species of the constituent solitons. These can either be two solitons of the same species (type A breathers) or solitons of anti species (type B breathers).

The topological charges of these breather solutions have been calculated. Type A breathers carry topological charges which lie in the fundamental representation  $\mathcal{R}_{\lambda_{2a \bmod h}} \subset \mathcal{R}_{\lambda_a} \otimes \mathcal{R}_{\lambda_a}$ , where  $a$  is the species of the constituent solitons. To be precise, these topological charges coincide with the topological charges of the single

soliton of species  $2a \bmod h$ . Therefore the fundamental representation  $\mathcal{R}_{\lambda_{2a \bmod h}}$  is normally not filled up [37]. It is a mystery that only certain combinations of the topological charges of the constituent solitons are allowed to bind together to yield a breather. An understanding of these phenomena is far from complete. It is conjectured that in the quantum theory there are more states than classical solutions [41]. In other words, the topological charges in the quantum theory have been conjectured to fill up the associated fundamental representation. As part of the spectrum of the quantum theory corresponds to classical soliton and breather solutions, one might have thought that the classical breather solutions give at least some of the missing topological charges. This appears not to be the case, at least for breathers with two constituent solitons.

Exceptional cases of type A breathers are those constructed from solitons of species  $(n + 1)$  in the  $a_{2n+1}^{(1)}$  theories. These are embedded sine-Gordon breathers which belong to type A and type B since both carry zero topological charge. They differ from the type B breathers as type B are sine-Gordon *like* breathers. It has been shown that in both of these cases, the topological charge lies in the *singlet* component of the Clebsch-Gordan decomposition of the tensor product of a fundamental representation with its conjugate representation.

Of no less important interest are breathers in other affine Toda theories. It would be interesting to know how many breathers can be constructed from their solitons. For the  $d_4^{(1)}$  theory this has been done in [101].



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## Chapter IV

### Discussion and Outlook

#### 4.1 Affine Toda Field Theories with Boundaries

The conditions for classical integrability of affine Toda field theories with a boundary are well established now [85-88]. For the quantum theory integrability has only been assumed following the ideas of Cherednik [71]. This has led to the discovery of reflection matrices for the quantum theory [79-84]. Unfortunately there seems to be a multitude of possible reflection matrices and their correspondence to particular boundary conditions is far from being well understood. It might be interesting to try to show explicitly, perhaps for a special case of affine Toda field theory first, that Cherednik's assumptions are indeed valid. It might be that this investigation would show that there are certain selection rules for the boundary conditions and the reflection matrices.

If that should prove too difficult one could try to adopt the perturbation scheme used by Kim [83,84] for the natural Neumann condition to work out reflection matrices for non-vanishing boundary conditions. This could be done in the  $a_2^{(2)}$  case where there is a continuous connection between the natural Neumann condition and non-zero boundary conditions for the classical integrability.

Another option for future research would be to perform numerical checks on the theory as Watts and Weston [66] have done for the mass ratios of a particular affine Toda field theory. The difficulty of this approach would be that it is not easy to reduce the error of numerical calculation easily to find evidence for two values to be the same. Still, it might give a surprising clue if the mass ratios for states in the boundary theory should turn out to be very different from the ones of the bulk theory.



## 4.2 Breathers in affine Toda Field Theories

After the successful calculation of breathers in the  $a_n^{(1)}$  theory [95] one should naturally look for breathers in other theories. One would expect them to exist and it would be interesting to see whether their topological charges behave in a similar manner to the ones in  $a_n^{(1)}$ . In part this has been done for the case of  $d_4^{(1)}$  in [101].

Because the search for breathers was motivated by the desire to find some of the “missing” topological charges in  $a_n^{(1)}$  and failed to provide them, one is tempted to review the situation and check whether there are actually missing solutions which have been overlooked. Recently Beggs and Johnson [107] published a preprint in which they claimed to have new solutions for  $a_n^{(1)}$ . The example they discuss,  $a_3^{(1)}$ , gives a valid  $\tau$  function but they are not well behaved because their energy momentum is not real. Also, their topological charges are not well defined. It is possible to show in a similar manner as in chapter 3 that the  $\tau$  functions pass through zero and are therefore not well behaved for any choice of parameters [108]. However to show this for the general case seems very difficult. But the fact that the example has faults casts doubts on the conjecture that this is a way to find new solutions.

One assumption has always been that the quantum theory would provide some explanation for the “missing” topological charges but so far this has not been the case. Breathers in affine Toda field theories however play an important part in the calculation of soliton  $S$ -matrices [42-44].

After the disappointment of not finding the charges one could perhaps try to find some arguments why they do not exist since the evidence of their non-existence seems strong.

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## Appendix A

### Lie Algebras

For the description of Toda theories a certain understanding of Lie-algebraic concepts is necessary. For affine Toda field theory infinite dimensional Lie algebras will be used. Still, it is easier to start with finite dimensional Lie algebras and generalise to certain infinite dimensional Lie algebras, Kač Moody algebras.

#### A.1 Finite Dimensional Lie algebras

A complete and rather exhaustive description of finite dimensional Lie algebras can be found in [109]. A *finite dimensional Lie algebra*  $\mathbf{L}$  is a vector space over  $\mathbb{R}$  or  $\mathbb{C}$  with a bilinear mapping  $[\cdot, \cdot] : \mathbf{L} \times \mathbf{L} \rightarrow \mathbf{L}$  called the commutator or bracket which satisfies the following conditions:

$$\forall a \in \mathbf{L} : [a, a] = 0 \quad (A.1a)$$

$$\forall a, b, c \in \mathbf{L} : [a, [b, c]] + [c, [a, b]] + [b, [c, a]] = 0. \quad (A.1b)$$

The relation (A.1a) together with the bilinearity implies that  $[x, y] = -[y, x]$  for all  $x, y \in \mathbf{L}$ . The condition (A.1b) is known as the Jacobi identity. The bracket  $[\cdot, \cdot]$  can usually be thought of as being the usual commutator for  $k \times k$  matrices,  $M, N \in M_k : [M, N] = MN - NM$ . But there are cases where it is, for instance, the Poisson bracket  $\{\cdot, \cdot\}$ , e.g. in classical mechanics for generalised momenta, coordinates and the Hamiltonian. By introducing a basis  $\{T_a\}$  for the Lie algebra  $\mathbf{L}$  much of its behaviour is encoded in the *structure constants*  $f_{ab}^c$ . They are given as the coefficients of the brackets expressed in terms of the basis elements

$$\forall T_a, T_b \in \{T_i\} : [T_a, T_b] = f_{ab}^c T_c.$$

The axioms (A.1) can be expressed in terms of structure constants.

A *sub-algebra*  $\mathbf{U}$  of a Lie algebra  $\mathbf{L}$  is a sub-vector space of  $\mathbf{L}$  which is closed under the bracket. If for a sub-algebra  $\mathbf{I}$ ,  $\mathbf{I} \subset \mathbf{L}$ , for all  $x \in \mathbf{L}$  and  $y \in \mathbf{I}$  the bracket of these two elements is in  $\mathbf{I}$ ,  $[x, y] \in \mathbf{I}$ ,  $\mathbf{I}$  is called an *ideal* of  $\mathbf{L}$ . The *derived*

algebra  $\mathbf{L}'$  of a Lie algebra  $\mathbf{L}$  is given by all linear combinations of brackets of  $\mathbf{L}$ ,  $\mathbf{L}' = \{x | \exists y, z \in \mathbf{L} : x = [y, z]\}$ .  $\mathbf{L}'$  is an ideal. A *simple Lie algebra* is a Lie algebra  $\mathbf{L}$  which has  $\{0\}$  and  $\mathbf{L}$  as its only ideals and the derived algebra  $\mathbf{L}'$  is not zero,  $\mathbf{L} \neq 0$ . For most applications and calculations involving Lie algebra one needs a representation of the Lie algebra. A *representation*  $\Phi$  of a Lie algebra  $\mathbf{L}$  is a homomorphism from  $\mathbf{L}$  to the endomorphism of a vector space  $V$ , i.e. the set of linear functions  $f : V \rightarrow V$ .

$$\Phi : \mathbf{L} \rightarrow \text{End}(V).$$

So for each element  $x \in \mathbf{L}$   $\Phi$  finds a linear mapping in  $V$ . This linear mapping can then be expressed in a basis of  $V$ . If  $V$  is finite dimensional there is a matrix  $M_\Phi$  for each  $\Phi$ . The dimension of  $V$  does not have to coincide with that of  $\mathbf{L}$ . If  $\Phi$  is a monomorphism the representation is called *faithful*. More generally representations are classified as *reducible* and *irreducible*, depending on whether or not there is a subspace  $W \subset V$  which is invariant under the action of the Lie algebra,  $W \notin \{\{0\}, V\}$ . An important rôle plays the *adjoint representation*. As  $\mathbf{L}$  itself is a vector space one can look at the representation

$$\text{ad} : \mathbf{L} \rightarrow \text{End}(\mathbf{L}).$$

The mapping is specified in terms of elements  $x \in \mathbf{L}$ :  $\text{adx} = [x, \cdot] \in \text{End}(\mathbf{L})$ . The image of the adjoint representation is the derived algebra, its kernel is the centre of  $\mathbf{L}$ . These linear functions can then be expressed as matrices with respect to a basis of  $\mathbf{L}$ . Often the set  $\{x | x \in \text{end}(\mathbf{L}), \exists y \in \mathbf{L} : x = [y, \cdot]\} \subset \text{End}(\mathbf{L})$  is called the adjoint representation of  $\mathbf{L}$ , though it is actually only  $\text{Im}(\text{ad})$ .

Also there is the *Killing form*  $K$  which is a symmetric bilinear form on  $\mathbf{L}$

$$K : \mathbf{L} \times \mathbf{L} \rightarrow \mathbb{C}, (x, y) \mapsto K(x, y) = \text{Tr}(\text{adx} \text{ady}).$$

The Killing form of a Lie algebra is non-degenerate, if and only if the Lie algebra is semi-simple. One way to check whether a given bilinear form is degenerate or not is to compute the determinant of the matrix in a particular basis. If and only if the form is degenerate the determinant will be zero. In the case of the Killing form the matrix is given in terms of the structure constants as

$$K_{ab} = c_{ade}c_{bed}.$$

A Lie algebra is called compact if the matrix for the Killing form is the negative of the unit matrix

$$\mathcal{M}(K) = -(\delta_{ij}).$$

This is only the case if the structure constants are totally antisymmetric.

The maximal Abelian sub-algebra of a Lie algebra is called the *Cartan sub-algebra*, its dimension is called the rank of the Lie algebra. A compact Lie algebra has a unique Cartan sub-algebra. Any Lie algebra can be decomposed into simple, compact Lie algebras

$$\mathbf{L} = \sum_{r=1}^p \bigoplus \mathbf{L}_r.$$

### A.1.1 Cartan decomposition

Assume  $\mathbf{L}$  is simple and find the Cartan sub-algebra  $H$  denoting a basis of it by  $\{H_i\}$   $i = 0, \dots, r$ . By definition all elements of the basis commute

$$[H_i, H_j] = 0 \quad \forall i, j \in \{0, 1, \dots, r\}. \quad (\text{A.2})$$

With these elements  $H_i$  find basis elements  $E_\alpha$  for the remaining algebra which obey

$$[H^i, E^\alpha] = \alpha^i E^\alpha \quad \alpha \in H^*. \quad (\text{A.3})$$

The element  $\alpha \in H^*$  is called a root. The set of all roots is denoted by  $\Phi$ . If  $\alpha$  is a root so is  $-\alpha$ . The step operator  $E^{-\alpha}$  corresponding to the negative root of  $\alpha$  is give by hermitian conjugation  $E^{-\alpha} = (E^\alpha)^\dagger$ . The commutation relation for the step operators are as follows:

$$[E^\alpha, E^\beta] = \begin{cases} \epsilon(\alpha, \beta) E^{\alpha+\beta} & \text{iff } \alpha + \beta \in \Phi \\ \frac{2\alpha \cdot H}{\alpha^2} & \text{iff } \alpha = -\beta \\ 0 & \text{otherwise.} \end{cases} \quad (\text{A.4})$$

A basis satisfying all relations (A.2) to (A.4) is known as a modified Cartan-Weyl basis. For each root  $\alpha \in \Phi$  there is an  $su(2)$  sub-algebra of  $\mathbf{L}$ , generated by  $E^\alpha$ ,  $E^{-\alpha}$  and  $\frac{2\alpha \cdot H}{\alpha^2}$ . This fact is used to show that  $\frac{2\alpha \cdot H}{\alpha^2}$  is an integer for any roots  $\alpha, \beta \in \Phi$ .

An important tool when dealing with roots is the Weyl-reflection. The Weyl-reflection  $r$  acts on roots as a permutation

$$r_\beta : \Phi \rightarrow \Phi, \quad \alpha \mapsto r_\beta(\alpha) = \alpha - \frac{2\beta\alpha}{\beta^2}.$$

The group generated by this transformation is called the Weyl-group. To classify all possible simple Lie algebras one has to take a linearly independent subset of all roots  $\Phi$ . This subset  $\Delta = \{\alpha_i\}$  is chosen in such a way that any root can be written as linear combination of the basis elements with integer coefficients

$$\alpha = \sum_{i=1}^r n_i \alpha_i \quad n_i \in \mathbb{Z}, \forall \alpha \in \Phi.$$

Actually all roots can be written with either the coefficient all being positive or negative. Hence they are called positive or negative roots, and the set of all roots can be written as sum of negative roots  $\Phi_-$  and positive roots  $\Phi_+$ ,  $\Phi = \Phi_- \cup \Phi_+$ . The *height* of a root is the sum of its coefficients

$$\text{ht}(\alpha) = \sum_{i=1}^r n_i.$$

Simple groups turn out to be completely determined by all scalar-products of their simple roots. Written as a matrix, known as Cartan matrix, this is

$$C_{ij} = \frac{2\alpha_i\alpha_j}{\alpha_j^2}.$$

The diagonal of this matrix is obviously 2 for all Cartan matrices. The common feature which all Cartan matrices share is that the off-diagonal entries are either zero or negative integers. An alternative way to describe a simple group is the Dynkin diagram. Each simple root is represented as a dot. Neighbouring roots are connected by  $C_{ij}C_{ij}$  lines. They correspond to the angle between the roots. An arrow points to the longer of two roots if they are of different length.

### A.1.2 Weight representations and the Chevalley basis

For any finite dimensional representation of a simple Lie algebra the action of the elements of the basis of the Cartan sub-algebra on the basis elements of the representation can be diagonalized

$$H|\mu\rangle = \mu|\mu\rangle.$$

The vector  $\mu = (\mu_1, \dots, \mu_r)$  is called the weight vector. It can be shown that the eigenvalue of  $\frac{2\alpha H}{\alpha^2}$  acting on  $|\mu\rangle$  is an integer. The set of all  $|\mu\rangle$  is called the weight lattice  $\Lambda_W(\mathbf{L})$ . The root lattice  $\Lambda_r(\mathbf{L})$  is a subset of the weight lattice. All  $\lambda_j \in \Lambda_W(\mathbf{L})$  satisfying  $\frac{2\alpha_i \lambda_j}{\alpha_i^2} = \delta_{ij}$  form a basis of the weight lattice. Any weight  $\lambda \in \Lambda_W(\mathbf{L})$  can be written as linear combination of this basis with integer coefficients  $\lambda = \sum_{i=1}^r \lambda_i n_i$ ,  $n_i \in \mathbb{Z}$ . If all coefficients  $n_i$  are positive the weight is called *dominant*. If  $\mu_0$  is a weight corresponding to the state  $|\mu_0\rangle$ , the differences  $\mu_0 - \mu$  for any other root  $\mu$  can be expressed as sum of roots. If further for all positive roots  $\alpha$ ,  $E^\alpha |\mu_0\rangle = 0$ , then this difference can be expressed using positive roots only. In this case  $\mu_0$  is called a highest weight. The weights of the adjoint representation are the roots. The highest weight is denoted as  $\Psi$  and its height is the Coxeter number  $h$ .

The Chevalley basis can be derived from the Cartan Weyl basis as follows. Set

$$e_\alpha = \sqrt{\frac{2}{\alpha^2}} \quad \text{and} \quad h_\alpha = \frac{2\alpha \cdot H}{\alpha^2}$$

For simplicity the quantities will be numbered by the index of the simple root

$$e_{\alpha_i} = e_i, \quad e_{-\alpha_i} = f_i, \quad \text{and} \quad h_{\alpha_i} = h_i.$$

The commutators for this basis turn out to be

$$[h_i, h_j] = 0, \quad [h_i, e_j] = C_{ji} e_j, \quad [h_i, f_j] = -C_{ji} f_j, \quad [e_i, f_j] = \delta_{ij} h_j.$$

Now a few words on gradings of algebras. One can define the following operator  $T^3$  which grades the Lie algebra  $\mathbf{L}$

$$T^3 = \frac{1}{2} \sum_{\Phi^+} \frac{2\alpha \cdot H}{\alpha^2}.$$

Grading means that  $[T^3, E^\alpha] = \text{ht}(\alpha) E^\alpha$ , or written in a multiplicative form with  $S = \exp(\frac{2\pi i T^3}{h})$

$$S E^\alpha S^{-1} = e^{\frac{2\pi i}{h} \text{ad} T^3} E^\alpha = e^{\frac{2\pi i}{h} \text{ht}(\alpha)} E^\alpha = \omega^{\text{ht}(\alpha)} E^\alpha$$

with  $\omega$  the  $h^{\text{th}}$  root of unity. So, the algebra exhibits a  $\mathbb{Z}_n$  grading

$$\mathbf{L} = \mathbf{L}_0 \oplus \dots \oplus \mathbf{L}_{n-1}.$$

The subscripts denote the height of the root associated with each operator. In particular  $\mathbf{L}_0$  is the Cartan sub algebra.

## A.2 Infinite dimensional Lie algebras

Untwisted affine Lie algebras are a particular sort of infinite dimensional Lie algebras. They are closely related to the simple Lie algebras discussed in the previous section. As for the simple Lie algebras they can be classified by Dynkin diagrams and Cartan matrices. These generalised Cartan matrices are those of the simple algebras with one column and row added. Similarly the Dynkin diagram has simply one dot added.

The algebras are described by introducing the concept of a loop algebra. For this take the algebra of all Laurent polynomials  $\mathcal{L} = \mathbb{C}[\lambda, \lambda^{-1}]$  with complex coefficients over  $\mathbb{C}$ . Define the following bilinear function

$$\begin{aligned}\phi : \mathcal{L} \times \mathcal{L} &\rightarrow \mathbb{C} \\ (P, Q) &\mapsto \phi(P, Q) = \text{Res}\left(\frac{dQ}{d\lambda}Q\right).\end{aligned}$$

With the Laurent polynomial written as

$$P = \sum_{n=-\infty}^{+\infty} c_n \lambda^n, \quad \text{Res}(P) = c_{-1}.$$

The function  $\phi$  fulfils the two following equations:

$$\begin{aligned}\phi(Q, P) &= -\phi(P, Q) \\ \phi(PQ, R) + \phi(QR, P) + \phi(RP, Q) &= 0 \quad \forall Q, P, R \in \mathcal{L}.\end{aligned}$$

This is used to define the loop algebra  $\mathcal{L}(\mathbf{L}) = \mathcal{L} \otimes_{\mathbb{C}} \mathfrak{bf} L$  with the bracket

$$\begin{aligned}[\cdot, \cdot] : \mathcal{L}(\mathbf{L}) \times \mathcal{L}(\mathbf{L}) &\rightarrow \mathcal{L}(\mathbf{L}) \\ (P \otimes l_1, Q \otimes l_2) &\mapsto [P \otimes l_1, Q \otimes l_2] = PQ \otimes [l_1, l_2].\end{aligned}$$

A bilinear  $\mathcal{L}$  on  $\mathcal{L}(\mathbf{L})$  is defined by

$$\begin{aligned}(\cdot | \cdot) : \mathcal{L}(\mathbf{L}) \times \mathcal{L}(\mathbf{L}) &\rightarrow \mathbb{C} \\ (P \otimes l_1, Q \otimes l_2) &\mapsto (P \otimes l_1, Q \otimes l_2) = PQ \otimes (l_1 | l_2).\end{aligned}$$

A derivation  $D$  on  $\mathcal{L}$  can be extended to one on  $\mathcal{L}(\mathbf{L})$  by

$$D(P \otimes l_1) = D(P) \otimes l_1.$$

A  $\mathbb{C}$ -valued cocycle  $\Psi$  on  $\mathcal{L}(\mathbf{L})$  is defined as

$$(a, b) \rightarrow \Psi(ab) = \text{Res}\left(\frac{da}{d\lambda} | b\right) = (l_1, l_2) \phi(P, Q) \quad a = P \otimes l_1, \quad b = Q \otimes l_2.$$

This functions has the following properties

$$\Psi(a, b) = -\Psi(b, a)$$

$$\Psi([a, b], c) + \Psi([b, c], a) + \Psi([c, a], b) = 0.$$

The one dimensional extension of  $\mathcal{L}(\mathbf{L})$  is  $\tilde{\mathcal{L}}(\mathbf{L}) = \mathcal{L}(\mathbf{L}) + \mathbb{C} K$  with  $[a + \mu K, b + \eta K] = [a, b] + \Psi(a, b)K$ .

Now, the affine Lie algebra  $\hat{\mathbf{L}}$  associated with the simple Lie algebra  $\mathbf{L}$  is given by adding a derivation  $d = \lambda \frac{d}{d\lambda}$  which provides a homogeneous grading of  $\hat{\mathbf{L}}$  with

$$dK = 0$$

$$[d, \lambda^m \otimes l_1] = m\lambda^m \otimes l_1 \quad \text{for } l_1 \in \mathcal{L}(\mathbf{L})$$

$$\hat{\mathbf{L}} = \tilde{\mathcal{L}}(\mathbf{L}) \oplus \mathbb{C} d = \mathcal{L}(\mathbf{L}) \oplus \mathbb{C} K \oplus \mathbb{C} d$$

where the last line is the decomposition of the Lie algebra. The bracket of  $\hat{\mathbf{L}}$  is given by

$$\begin{aligned} &[(\lambda^m \otimes \mathbf{L}_1) \oplus \mu_1 K \oplus \eta_1 d, (\lambda^n \otimes \mathbf{L}_2) \oplus \mu_2 K \oplus \eta_2 d] = \\ &(\lambda^{m+n} \otimes [\mathbf{L}_1, \mathbf{L}_2] + \eta_1 \lambda^n \otimes \mathbf{L}_2 - \eta_2 m \lambda^m \otimes \mathbf{L}_1 \oplus m \delta m, -n(\mathbf{L}_1 \mathbf{L}_2)K). \end{aligned}$$

The Cartan sub algebra  $\hat{h}$  of  $\hat{\mathbf{L}}$  is  $r + 2$  dimensional if  $r = \text{rank } \mathbf{L}$  and

$$\hat{h} = h \oplus \mathbb{C} K \oplus \mathbb{C} d.$$

Its Chevalley basis is

$$h_i = \lambda^0 \otimes H^{\alpha_i}, \quad e_i = \lambda^0 \otimes E^{\alpha_i}, \quad f_i = \lambda^0 \otimes E^{-\alpha_i} \quad i \neq 0,$$

$$h_0 = \lambda^0 \otimes H^{-\Psi}, \quad e_0 = \lambda^1 \otimes E^{-\Psi}, \quad f_0 = \lambda^{-1} \otimes E^{\Psi}$$



in terms of the Chevalley generators of  $\mathbf{L}$ . So, the new Chevalley generators satisfy

$$[h_i, h_j] = 0, [h_i, e_j] = K_{ji}e_j, [h_i, f_j] = -K_{ji}f_j, [e_i, f_j] = \delta_{ij}h_j$$

and determine the algebra together with the Serre relation

$$(\text{ad } e_i)^{1-K_{ij}}e_j = 0, (\text{ad } f_i)^{1-K_{ij}}f_j = 0, \quad i \neq j.$$

$K$  is defined as the extended Cartan matrix of the algebra  $\hat{\mathbf{L}}$  defined like the one for  $\mathbf{L}$  but with an additional column and row for  $\Psi = -\alpha_0$ . A complete list of Dynkin diagrams can be found in Kač's book [46].

### A.2.1 Grading, Coxeter Numbers and Weight Representations

When expressed in the Chevalley generators the algebra  $\hat{\mathbf{L}}$  exhibits a grading structure with the element  $d' = dh + \lambda_0 \otimes T^3$

$$[d', h_i] = 0, [d', e_i] = e_i, [d', f_i] = -f_i$$

where  $h_i$  and  $d_i$  span the Cartan sub algebra. Define the smallest set of natural numbers  $m_i, n_i$  for which

$$\sum_i K_{ij}m_i = 0 \quad \text{and} \quad \sum_j K_{ij}n_j = 0$$

then  $n_i = \frac{\Psi^2 m_i}{\alpha_i^2}$ . The Coxeter number and the dual Coxeter number are defined as  $h = \sum_i n_i$  and  $\bar{h} = \sum_i m_i$  respectively. The element  $x = \frac{2k}{\Psi^2} = \frac{2k}{\sum_i m_i n_i}$  is central. If the central extension of  $\hat{\mathbf{L}}$  is non zero there is a highest weight representation. These representations are formed by a highest weight state  $|\Lambda\rangle$  acted on by an arbitrary number of negative step operators. The highest weight state is characterised by the action of  $h_i$

$$h_i|\Lambda\rangle = \Lambda(h_i)|\Lambda\rangle = \frac{2\Lambda a_i}{a_i^2}|\Lambda\rangle.$$

The fundamental weights  $\Lambda_j$  generate the weight lattice  $\Lambda_W$

$$\frac{2\Lambda_j \cdot a_i}{a_i^2} = \delta_{ij}.$$

The root space is similar to that of the corresponding non-affine Lie algebra with extra dimensions for  $k$  and  $d$

$$a_i = (\alpha_i, 0, 0) \ i \neq 0 \quad a_0 = (-\Psi, 0, 0).$$

The definition of the inner product is

$$(\beta_1, c_1, d_1) \cdot (\beta_2, c_2, d_2) = \beta_1\beta_2 + c_1d_2 + c_2d_1.$$

With an arbitrary final component the fundamental weights take the form

$$\Lambda_i = (\lambda_i, m_i\Psi^2/2, 0) \ i \neq 0, \quad \Lambda_0 = (0, \Psi^2/2, 0).$$

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